PRIME LABELING OF HEAWOOD GRAPHS

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Abstract: A graph with vertex set $V$ is said to have a prime labeling if its vertices are labeled with distinct integers $1, 2, \ldots |V|$ such that for edge $xy$ the labels assigned to $x$ and $y$ are relatively prime. A graph which admits prime labeling is called a prime graph. In this paper I investigate prime labeling for some special graph namely Heawood graph. I also discuss prime labeling in the context of graph operations namely duplication and switching in Heawood graph.

Keywords: Prime labeling, Heawood graph, Duplication, Switching.

I. INTRODUCTION

In this paper, I consider only finite simple undirected graph. The graph $G$ has vertex set $V = V(G)$ and edge set $E = E(G)$. The set of vertices adjacent to a vertex $u$ of $G$ is denoted by $N(u)$. For notations and terminology we refer to Bondy and Murthy [1].

The notion of prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout [6]. Two integers $a$ and $b$ are said to be relatively prime if their greatest common divisor is 1. Many researchers have studied prime graph. Fu.H[3] has proved that the path $P_n$ on $n$ vertices is a prime graph. Deretsky et al [2] have proved that the cycle $C_n$ on $n$ vertices is a prime graph. Around 1980 Roger Entringer conjectured that all trees have prime labeling which is not settled today.

In [7] S.K. Vaidhya and K.K. Kanmani have proved that the graphs obtained by identifying any two vertices duplicating arbitrary vertex and switching of any vertex in cycle $C_n$ admit prime

The following definitions are necessary for the present study.

**Definition 1.1**

Let $G = (V(G), E(G))$ be a graph with $p$ vertices. A bijection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ is called a prime labeling if for each edge $e = uv$, $\gcd(f(u), f(v)) = 1$. A graph which admits prime labeling is called a prime graph.

**Definition 1.2**

Duplication of a vertex $v$ of graph $G$ produces a new graph $G'$ by adding a new vertex $v'$ such that $N(v') = N(v)$. In other words a vertex $v'$ is said to be duplication of $v$ if all the vertices which are adjacent to $v$ in $G$ are also adjacent to $v'$ in $G'$.

**Definition 1.3**

Duplication of a vertex $v_k$ by a new edge $e = v_k' v_k'$ in a graph $G$ produces a new graph $G'$ such that $N(v_k') = \{v_k, v_k', v_k\}$ and $N(v_k') = \{v_k, v_k', v_k\}$.

**Definition 1.4**

A vertex switching $G_v$ of a graph $G$ is obtained by taking a vertex $v$ of $G$, removing the entire edges incident with $v$ and adding edges joining $v$ to every vertex which are not adjacent to $v$ in $G$.

**Definition 1.5**

A regular graph is a graph where every vertex has the same degree. A regular graph with vertices of degree $k$ is called a $k$-regular graph.

**III. HEAWOOD GRAPH**

**Definition 1.6**

The Heawood graph is an undirected graph with 14 vertices and 21 edges. Heawood graph is a 3-regular graph.
Example

IV. MAIN RESULTS

Theorem 3.1

Heawood graph is a prime graph.

Proof:

Let G be the heawood graph with 14 vertices and 21 edges.

Then \(|V(G)| = 14\) and \(|E(G)| = 21\).

The edge set \(E(G) = \{v_i v_{i+1} / 1 \leq i \leq 13\} \cup \{v_1 v_{14}\} \cup \{v_i v_{i+5} / i = 1,3,5,7,9\}

\cup \{v_2 v_{11}\} \cup \{v_4 v_{13}\}\).

Define a labeling \(f. \ V(G) \to \{1,2, \ldots \ldots ,14\}\)

by \(f(v_i) = i\), \(i = 2,3,4,6,7,8,9,10,11,12,13,14\).

\(f(v_1) = 5\)

\(f(v_5) = 1\)

Then \(g.c.d \{f(v_i), f(v_{i+1})\} = 1, \quad 1 \leq i \leq 13\)

\(g.c.d \{f(v_i), f(v_{i+5})\} = 1, \quad i = 1,3,5,7,9,\)

\(g.c.d \{f(v_1), f(v_{14})\} = 1\)

\(g.c.d \{f(v_2), f(v_{11})\} = 1\)

\(g.c.d \{f(v_4), f(v_{13})\} = 1\)
Then f admits a prime labeling. Hence G is a prime graph.

**Illustration of Theorem 3.1**

![Illustration of Theorem 3.1](image)

**Theorem 3.2**

Duplication of a vertex of Heawood graph is a prime graph.

**Proof:**

Let G be the Heawood graph.
Then \(|V(G)| = 14\) and \(|E(G)| = 21\).
Let \(G^*\) be the graph obtained by duplication of a vertex \(v_3\) of Heawood graph.
Let the new vertex be \(v'_3\).
Then \(|V(G^*)| = 15\) and \(|E(G^*)| = 24\).
The edge set \(E(G^*) = \{v_i v_{i+1}/1 \leq i \leq 13\} \cup \{v_i v_{14}\} \cup \{v_i v_{i+5}/i = 1, 3, 5, 7, 9\} \cup \{v_2 v_{11}\} \cup \{v_4 v_{13}\} \cup \{v_2 v'_3\} \cup \{v_4 v'_3\} \cup \{v_8 v_3\}\).
Define a labeling \(f: V(G^*) \rightarrow \{1,2, \ldots, 15\}\) by
\[
f(v_i) = i, \quad i = 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14.
\]
\[
f(v_1) = 5
\]
Then $f$ admits Prime labeling. Hence $G^*$ is a prime graph.

**Illustration of Theorem 3.2**

Prime labeling for duplication of a vertex $v_3$ in Heawood graph

**Theorem 3.3**

Duplication of an edge by a vertex of Heawood graph is a prime graph.

**Proof:**

Let $G$ be the Heawood graph. Then $|V(G)| = 14$ and $|E(G)| = 21$.

Let $G^*$ be the graph obtained by duplication of an edge by a new vertex $w$ of Heawood graph. Then $|V(G^*)| = 15$ and $|E(G^*)| = 23$.

The edge $E(G) = \{v_i v_{i+1} / 1 \leq i \leq 13\} \cup \{v_1 v_{14}\} \cup \{v_1 v_{i+5} / i = 1, 3, 5, 7, 9\} \cup \{v_2 v_{11}\} \cup \{v_4 v_{13}\} \cup \{v_i v_w : i = 4, 5\}$

Define a labeling $f: V(G^*) \rightarrow \{1, 2, \ldots, 15\}$

by $f(v_i) = i, i = 2, 3, 4, 6 \ldots \ldots \ldots 14$
\[
f(v_1) = 5 ; \ f(v_5) = 1 ; \ f(\omega) = 15.
\]

Then \(f\) admits prime labeling. Hence \(G^*\) is a prime graph.

**Illustration of theorem 3.3**

![Diagram](image)

Prime labeling for duplication of an edge \(v_4v_5\) by a vertex \(w\) in Heawood graph

**Theorem 3.4**

Let \(G^*\) be the graph obtained by switching a vertex in the Heawood graph. Then \(G^*\) is a Prime graph.

**Proof:**

Let \(G\) be the Heawood graph.

Then \(|V(G)| = 14\) and \(|E(G)| = 21\).

Let \(G^*\) be the graph obtained by switching a vertex \(v_5\) of Heawood graph.

Then \(|V(G^*)|=14\) and \(|E(G^*)|=28\).

The edge \(E(G^*) = \{v_i v_{i+1} : i = 1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13\} \cup \{v_1 v_{14}\} \cup \{v_i v_{i+5} : i = 1, 3, 7, 9\} \cup \{v_2 v_{11}\} \cup \{v_4 v_{13}\} \cup \{v_5 v_1 : i = 1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14\}\)

Define a labeling \(f : V(G^*) \to \{1, 2, \ldots, 14\}\)

by \(f(v_i) = i\), \(i = 2, 3, 4, 6, 7 \ldots\) \(14\)

\[f(v_1) = 5\]

\[f(v_5) = 1\]

Then \(f\) admits prime labeling. Hence \(G^*\) is a prime graph.
Illustration of theorem 3.4

Primes labeling for switching of a vertex \( v_5 \) in Heawood graph

**Theorem 3.5**

Let \( G^* \) be the graph obtained by adjoining a vertex to each vertex in the Heawood graph. Then \( G^* \) is a prime graph.

**proof:**

Let \( G \) be the Heawood graph. Then \( |V(G)| = 14 \) and \( |E(G)| = 21 \).

Let \( G^* \) be the graph obtained by adjoining a vertex to each vertex in the Heawood graph.

Let \( w_1, w_2, \ldots, w_{14} \) be the vertices adjoining to each vertex in the Heawood graph.

Then \( |V(G^*)| = 28 \) and \( |E(G^*)| = 35 \).

The edge set \( E(G^*) = \{ v_i v_{i+1} / 1 \leq i \leq 13 \} \cup \{ v_1 v_{14} \} \cup \{ v_i v_{i+5} / i = 1, 3, 5, 7, 9 \} \)

\[ \cup \{ v_2 v_{11} \} \cup \{ v_4 v_{13} \} \cup \{ v_{i+1} w_i : i = 1 \leq i \leq 13 \}. \]

Define a labeling \( f : V(G) \to \{1, 2, \ldots, 28\} \)

by \( f(v_i) = i \), \( i = 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14. \)

\( f(v_1) = 5, \)

\( f(v_5) = 1, \)

\( f(w_i) = 14 + i \) : \( i = 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14. \)

\( f(w_4) = 26, \)

\( f(w_{12}) = 18. \)
Then f admits a Prime labeling. Then $G^*$ is a Prime Graph.

**Illustration of Theorem 3.5**

![Diagram of prime labeling for adjoining of a vertex to each vertex in the Heawood graph](image)

prime labeling for adjoining of a vertex to each vertex in the Heawood graph

**V. CONCLUSION**

In the present work I investigate prime labeling of the special graph namely Heawood graph. To investigate similar results for other graph families and in the context of different graph operations is an open area of research.
VI. ACKNOWLEDGEMENTS

The author is thankful to the referee for the valuable comments which improves the standard of the paper.

REFERENCES


