COTOTAL BLOCK DOMINATION IN GRAPHS

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ABSTRACT:
For any graph $G(V, E)$, block graph $B(G)$ is a graph whose set of vertices is the union of the set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent . A dominating set D of a graph $B(G)$ is a cototal block dominating set if the induced subgraph $\langle V[B(G)] - D \rangle$ has no isolated vertices.The cototal block domination number $\gamma_{bct}(G)$ is the minimum cardinality of a co total block dominating set of G. In this paper many bounds on $\gamma_{bct}(G)$ are obtained interms of elements of G but not the elements of $B(G)$. Also its relation with other domination parameters were established.

Key words: Dominating set/Block graphs/co total block domination

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Introduction:

All graphs considered here are simple, finite, nontrivial, undirected, connected without loops or multiple edges. As usual, $p\ and\ q$ denote the number of vertices and edges of a graph G. For any undefined term or notation in this paper can be found in $Harary$ [2] .A set D of a graph G is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G) of\ G$ is the minimum cardinality of a dominating set.A dominating set D of a graph G is a cototal dominating set of $G. If\ every\ v \in V-D$ is not an isolated vertex in the induced subgraph $\langle V-D\rangle$. The cototal domination number $\gamma_{ct}(G) of\ G$ is the minimum cardinality of a co total dominating set. The concept was introduced by $Kulli$[4]. Now we define cototal block domination in graphs. A dominating set $D\ of\ B(G)$ is a cototal dominating set if the induced sub graph $\langle V[B(G)]-D\rangle$ has no isolated vertices the cototal block domination number $\gamma_{bct}(G)$of B(G) is the minimum cardinality of a cototal block domination set. As usual, the minimum degree of a vertex in G is denoted by $\delta(G)$.A vertex v is called a cut vertex if removing it from G increases the number of components of G. For any real number x, $\lceil x\rceil$ denotes the smallest integer not less than x.Aset of vertices in a graph G is called an independent set if no two vertices in the same set are adjacent. The vertex independence number $\beta_0(G)$ is the maximum cardinality of an independent set of vertices. A dominating set D is a total dominating set if the induced subgraph $\langle D\rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set.This concept was introduced by C.J.Cockayne[1]. A dominating set D is a connected dominating set whose induced sub graph $\langle D\rangle$ is connected. This concept was introduced by E.Sampath Kumar [7]. A dominating set D of a graph $G=(V,E)$is a non split dominating set if the induced sub graph $\langle V-D\rangle$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ of a graph G is the minimum cardinality of a non split dominating set. This concept was introduced by $Kulli$ [5].

In this paper many bounds on $\gamma_{bct}(G)$are obtained in terms of elements of G but not the elements of $B(G)$, also its relation with other domination parameter is established.

We need the following Theorems for our further results.

Theorem A[3] : A cototal dominating set $D\ of\ G$ is minimal if and only if for a vertex $v \in D$, one of the following conditions holds.

i) There exists a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$

ii v is an isolated vertex in $\langle D \rangle$

iii) v is an isolated vertx in $\langle (V - D) \cup \{v\} \rangle$

Theorem B[6]: If G is a graph with no isolated vertices then $\gamma(G) \leq \frac{p}{2}$.

Theorem 1: For any graph G with $n - blocks$ and $B(G) \neq K_2 and\ K_{1,p}\ p \geq 3\ then$

$$\gamma_{bct}(G) \leq n - 2$$

Proof : Suppose $B(G)$ be a block graph of a graph G .Let $H = \{B_1, B_{2,} B_3, \dots \dots \dots B_n\}$ be the blocks of G and $H_1 = \{b_{1,} b_{2,} b_3, \dots \dots \dots\ b_n\}$ be the set of vertices of $B(G)$ which corresponds to the blocks of H. Now we consider the following cases.

$case1$: Suppose every cut vertex of G lies on atleast three blocks. Let $D_1 = \{b_i\}\ 1 \leq i \leq n$ set of cut vertices which are incident to the end blocks of $B(G)$. Again we consider the set $D_2 = \{b_s\}, 1 \leq s \leq n\ \forall\ \mathrm{b_s} \notin \mathrm{N(D_1)}$. since $\langle \mathrm{V[B(G)]} - \{\mathrm{D_1} \cup \mathrm{D_2}\} \rangle$ does not have an isolated vertices.Then $D_1 \cup D_2$ is a minimal cototal dominating set in$B(G)$. $clearly$ $|D_1 \cup D_2| = \gamma_{bct}(G)$ which gives $\gamma_{bct}(G) \leq n - 2$.

$case2$: Suppose every cut vertex of G lies on atmost two blocks of G and atleast one nonend block is adjacent with atleast three blocks. Then $B(G)$ is a tree.Further we consider the two sub cases of $case2$

$subcase$ 2.1 : Assume $B(G)$ is a tree with $\Delta[B(G)] \geq 3$. Let $D_1{}^1 = \{b_i\}, 1 \leq i \leq n$ be the set of all end vertices in $B(G)$. Suppose $\exists\ b_k \in B(G)$ is an end vertex and if the distance from b_k to the nearest vertex with degree ≥ 3 is atleast four , then $b_k \in D_2$ and $K = \{b_{1,} b_{2,} b_3, \dots \dots \dots\ b_s\}$ where $\forall\ b_{i\,,} 1 \leq i \leq s$ are the vertices such that the distance between them is 3 with degree $b_i = 2$. Then $b_k \cup K$ gives the minimal block cototal dominating set. If there exists a path less than$H$, then b_k and $N(b_k) \in D_2$. $Hence$ $\left| D_1{}^1 \cup D_2 \right|$ is a minimal block cototal domination set of $B(G)$. Clearly $\left|D_1{}^1 \cup D_2\right| \leq \mathrm{n} - 2$ which gives $\gamma_{bct}(G) \leq n - 2$

$subcase 2.2$: Assume $B(G)$ is a tree with $\Delta[B(G)] \leq 2$ then $B(G)$ is a path. Let $B(G) = P_n: b_1, b_2, b_3, \dots \dots \dots b_n$ be a path. Now $D_1 = \{b_1, b_4, \dots \dots \dots b_{n-2}, b_{n-1}, b_n\}$. $If\ P_n$ consists of $6k$ number of vertices for $K = 1,2,3 \dots \dots \dots$ then $D = \{b_1, b_4, \dots \dots \dots b_{n-2}, b_{n-1}, b_n\}$ be the minimal cototal dominating set of $B(G)$. clearly $|D| = \gamma_{bct}(G) \leq n - 2$.

$If\ P_n$ Consists of other than $6k$ number of vertices, then the block cototal dominating set

$D = \{b_1, b_4, b_8 \dots \dots \dots b_n\}$ .Since each edge is a block in G with $n - 1$ number.Then $B(G)$ has $n - 2\ blocks$ .Clearly D gives the minimal block cototal dominating set and $n - 2 \geq |D|$ which gives $\gamma_{bct}(G) \leq n - 2$.

Theorem 2: For any graph G , $B(G) \neq K_2\ or\ K_{1,n}$, $n \geq 3$ then $\gamma_{bct}(G) \leq \gamma_{cot}(G)$

$Proof$: Suppose $B(G) = K_2\ or\ K_{1,n}\ n \geq 3$. Then cototal dominating set dose not exists for $B(G)$.Hence $B(G) \neq K_2 or K_{1,n}\ n \geq 3$. To establish the upperbound for $\gamma_{bct}(G)$, we have the following cases.

$case 1$: Suppose G has atleast one block which is not an edge.Then there exists atleast one block which contains more than one vertex. Let $V(G) = \{v_1, v_2, v_3, \dots \dots \dots v_n\}$ and Suppose, $\exists\ B_i$ blocks in $G, i \geq 2$ with more than two vertices. Let $D^1 \subset V(G)$ such that $D^1 = \{V_j\}, 1 \leq j \leq n$ be a cototal dominating set of G. Suppose there exists some vertices of D^1 with $j \geq 3 \in B_i$ in G. Hence $|D^1| = \gamma_{cot}(G)$. Let $H = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the set of blocks of G.Then there exists $H^1 = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the set of vertices in $B(G)$ corresponding to the blocks of H. Assume some $B_i \in H$ have more than two vertices in G.Then the corresponding $b_i \in H^1$ have a single-tone in H^1. Now we consider $D \subseteq H^1$ which is a cototal dominating set of $B(G)$. If all $b_i{}'s$ belongs to $D, then\ |D| = \gamma_{bct}(G) \leq |D^1|$ which gives $\gamma_{bct}(G) \leq \gamma_{cot}(G)$.

$case 2$: Suppose each block of G is an edge. Then G is a tree with $V(G) = \{v_1, v_2, v_3, \dots \dots \dots v_p\}$. Let $B_1 = \{v_i\}$, $1 \leq i \leq p$ such that $B_1 \subseteq V(G)$ and every v_i is an end vertex, $B_2 = \{v_j\}$,

$B_2 \subseteq V(G)$ each v_j is a vertex whose neighbour form an edge in a cototal dominating set of G. Now $D_1 = B_1 \cup B_2$ is a co total dominating set of G.Then $|D_1| = \gamma_{cot}(G)$. Suppose

$H = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the blocks of G .Then $H^1 = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the corresponding block vertices in $B(G)$. we consider the non end blocks of G which are cut vertices in $B(G)$. Let $H_1 = \{B_k\}$ be the set of all non end blocks of G which gives ${H_1}^1 = \{b_K\}$ be a set of cut vertices in $B(G)$. Hence ${H_1}^1$ is a $\gamma_{bct} - set, {H_1}^1 = \gamma_{bct}(G)\ clearly\ |{H_1}^1| \leq |D_1|$ which gives $\gamma_{bct}(G) \leq \gamma_{cot}(G)$

Theorem 3: For any $(p, q) graph\ G, with\ m$ end blocks,$B(G) \neq K_2\ or\ K_{1,n}\ n \geq 3$

$then\ \gamma_{bct}(G) \leq p - m$.

$Proof$: Suppose $B(G)$ is a complete graph $K_2\ or\ K_{1,n}$, $n \geq 3$, by definition of cototal block domination the result does not exists. Hence $B(G) \neq K_2 and\ K_{1,n}$, $n \geq 3$.

For establishing upper bound to $\gamma_{bct}\ (G)$. Suppose $S = \{B_1 , B_2 , B_3 , \dots \dots \dots B_n \}$ be the blocks of G and $M = \{b_1 , b_2 , b_3 , \dots \dots \dots . b_n\}$ be the block vertices in $B(G)$ corresponding to the blocks of G. Now $M_1 = \{b_1 , b_2 , b_3 , \dots \dots \dots . b_m \}$ $1 \leq m \leq n$, $M_1 \subset$ M be the set of all end vertices in $B(G)$. Let $J = \{b_1 , b_2 , b_3 , \dots \dots \dots . b_s\}$ be the set of all cut vertices in $B(G)\ and$ consider $J_1 \subseteq J$ such that $J_1 \neq \emptyset$ .Now $\langle M[B(G)] - (M_1 \cup J_1)\rangle$ has no isolated vertex which gives a co - total block dominating set in $B(G). Hence\ |M_1 \cup J_1| = \gamma_{bct}\ (G)$. Clearly $|M_1 \cup J_1| \leq |P| - |m|$ which gives $\gamma_{bct}\ (G) \leq P - m.$

Suppose , $J_1 = \emptyset$ and every non end vertex has atleast two vertices which are adjacent with other cut vertices $. Then\ \langle M[B(G)] - M_1\rangle$ has no isolates which gives a cototal block dominating set. Hence $|M_1| = \gamma_{bct}\ (G)$. Clearly $|M_1| \leq |P| - |m|$ and is $\gamma_{bct}\ (G) \leq P - m.$

Theorem 4 : For any graph $G\ with\ B(G) \neq K_2 and\ K_{1,n}, n \geq 3\ then\ \gamma_{bct}(G) \leq P - \delta(G) - 2$

$Proof$: Suppose $B(G)$ be a block graph of a graph G.Let $H = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the set of all blocks in $G\ and\ H^1 = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the vertices of $B(G)$ corresponding to the blocks of H. Let v be the vertex of minimum degree $\delta(G)$ such that $1 \leq \delta(G) \leq P - 1$. we have the following cases.

$case1$: Suppose $\delta(G) = 1$ we consider the following subcases of case1.

$subcase1.1$: Assume that each block is an edge then $q = p - 1$ which gives $n = p - 1\ or$

$n-2=p-3$ $by\ Theorem\ 1$ $,\gamma_{bct}(G)=p-\delta(G)-2.$

$subcase$1.2: Assume that there exists atleast one block which is not an edge. Then $n<p-1$

Which gives $n-2<p-3.By\ Theorem\ 1,\ \gamma_{bct}(G)<p-\delta(G)-2.$

On combining these two subcases, we have $\gamma_{bct}(G)\leq p-\delta(G)-2.$

$case2$: Suppose $\delta(G)\geq 2$.Then each block is not an edge.If G contains atleast $n-blocks$ and each block consists of at least three vertices, then G contains atleast $3n\ vertices$. Therefore

$p-\delta(G)-2\geq 3n-2\geq n-2\geq\gamma_{bct}(G).Hence\ \gamma_{bct}(G)\leq p-\delta(G)-2.$

Theorem 5: For any graph $G(p,q)$, $B(G)\neq K_2\ and\ K_{1,n}, n\geq 3\ then\ \gamma_{bct}(G)\leq\left\lceil\frac{p}{2}\right\rceil$

$Proof$: Suppose $B(G)$ is a complete graph $K_2\ or\ K_{1,n}$ $,n\geq 3$. Then by definition cototal block domination does not exists. Hence $B(G)\neq K_2\ and\ K_{1,n}$ $,n\geq 3$.

$For\ establishing\ upperbound\ to\ \gamma_{bct}$ $,we\ have\ the\ following\ cases.$

$case$ 1: Suppose each block of G is an edge.Then G is a tree.Let $S=\{B_1,B_2,B_3,\dots\dots\dots B_n\}$ be the blocks of G and $M=\{b_1,b_2,b_3,\dots\dots\dots b_n\}$ be the block vertices in $B(G)$ corresponding to the blocks $B_1,B_2,B_3,\dots\dots\dots B_n$ $of\ S.Let\ S_1=\{B_i\}$,$1\leq i\leq n$ be the set of all nonend blocks in$G$ and are cut vertices in $B(G)$. Again we consider a subset $S_2=\{B_j\},1\leq j\leq n\ of\ S$ such that the set $\{B_j\}$ is a set of all end blocks in G . Let $M_1=\{b_i\}$ be the set of all block vertices with respect to S_1, which are cut vertices in $B(G)$ and $M_2=\{B_j\}$ be the set of all non cut vertices corresponding to the setS_2 $in\ B(G).Let\ M_1{}^1\subseteq M_1\ and\ M_2{}^1\subseteq M_2$ .Now V[$B(G)$] $=S$ $,\forall\ v_i\in S-\{M_1{}^1\cup M_2{}^1\}$ has at least degree one. Then $\langle S-\{M_1{}^1\cup M_2{}^1\}\rangle$ has no isolates. Hence$|M_1{}^1\cup M_2{}^1|=\gamma_{bct}$ .But $\gamma_{bct}\leq min\{|M_1{}^1\cup M_2{}^1|,|S-(M_1{}^1\cup M_2{}^1)|\}$ by Theorem (B) $\gamma_{bct}(G)\leq\left\lceil\frac{p}{2}\right\rceil$

$case2$: Suppose $G\ has\ atleast$ one block which is not an edge. Let $S=\{B_1,B_2,B_3,\dots\dots\dots B_n\}$

be the blocks of G and $M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the block vertices in $B(G)$ corresponding to the blocks $B_1, B_2, B_3, \dots \dots \dots B_n$ of S. Assume some $B_i \in S$ have more than two vertices in$G$ .Then there exists atleast one block B_{i} ,$1 \leq i \leq n$ $such$ $that$ $V[B_i] \geq 2$.Let M_1 be a set of all cut vertices, M_2 is the set of all non cut vertices in $B(G)$ such that $M_1, M_2 \subseteq M$. Now we consider $M_1{}^1 \subseteq M_1$. Suppose $M_2 = \emptyset$ in $B(G)$.$Then$ $\langle V[B(G)] - \{M_1{}^1\}\rangle$ has no isolate and

$|M_1{}^1| = \gamma_{bct}(G)$. Suppose $M_2 \neq \emptyset$ in$B(G)$.Then there exist a subset $M_2{}^1 \subseteq M_2$ such that

$\langle V[B(G)] - \{(M_1{}^1 \cup M_2{}^1\}\rangle$ gives no isolate. Clearly $|(M_1{}^1 \cup M_2{}^1)| = \gamma_{bct}(G)$.$Since$ as in $case$1, we have $|M_1{}^1|$ or $|(M_1{}^1 \cup M_2{}^1| \leq \left|\frac{V[B(G)]}{2}\right|$ $which$ $gives$ $\gamma_{bct}(G) \leq \left[\frac{p}{2}\right]$.

Theorem 6: For any graph G, B(G)$\neq K_2$ and $K_{1,n}$,$n \geq 3$ $then$ $\gamma_{bct}(G) \leq S$,$where$ S is the number of cut vertices in G.

$Proof$: Suppose $B(G)$ be a block graph of a graph G. If $B(G)$ is either $K_{1,n}$ or a complete graph K_2 .Then by definition, cototal block domination does not exists. Hence $B(G) \neq K_2$ and

$K_{1,n}, n \geq 3$.

Suppose $M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the block vertices in $B(G)$ corresponding to the blocks of G .Let $M_1 = \{b_1, b_2, b_3, \dots \dots \dots b_j\} \subseteq$ M where $1 \leq j \leq n$ be the set of all end vertices in $B(G)$.Also $M_2 = \{b_1, b_2, b_3, \dots \dots \dots b_i\} \subset$ M ,$1 \leq$ i $\leq$ n be the set of all cut vertices in $B(G)$. Further we consider a set $M_3 = \{b_1, b_2, b_3, \dots \dots \dots b_s\}$ $1 \leq$ s $\leq$ i such that $M_3 \subset M_2$.

Now $\langle M[B(G)] - (M_1 \cup M_3)\rangle$ has no isolated vertices which gives a co-total block domination in $B(G)$. Hence $|M_1 \cup M_3| = \gamma_{bct}(G)$. Suppose every non end block has at least two blocks which are adjacent with different cut vertices and is denoted these cut vertices by a set S. Then by the definition of $B(G)$, $|S| \geq |M_1 \cup M_3|$ which gives $\gamma_{bct}(G) \leq S$.

On observing all the results connected to cototal block domination, we have easily obtain the following

Corollary 1: For a tree $T, B(G) \neq K_2$ and $K_{1,n}$,$n \geq 3$ $then$ $p - q \leq \gamma_{bct}(G)$

Corollary 2: For any graph $G, B(G) \neq K_2$ and $K_{1,n}$,$n \geq 3$ then $\gamma_{bct}(G) = \gamma(G)$

$if and\ only\ if\ G\ is\ a\ star.$

Theorem 7: For any graph $G, B(G) \neq K_2\ or K_{1,n}\ , n \geq 3\ then\ \ \gamma_{bct}\ (G) \leq P - \gamma_t(G)$

$Proof$: By the definition of cototal domination, $B(G) \neq K_2\ or\ K\ _{1,n} n \geq 3$. we consider the following cases.

$case$ 1 : Assume G is a tree and let $S = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the blocks of G and $M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the block vertices in $B(G)$ corresponding to blocks $B_1, B_2, B_3, \dots \dots \dots B_n\ of\ S. Let\ \{B_i\} \subseteq S$ such that all $B_i's$ are non-end blocks of G .Then $\{b_i\} \subseteq V[B(G)]$ which are cut vertices corresponding the set$\{B_i\}$. Since each block is complete in $B(G)$. Then every vertex of $V[B(G)] - \{b_i\}$ is adjacent to at least one vertex of $\{b_i\}$. Clearly $|b_i| = \gamma_{bct}(G)$. Since for a tree $T, P = q + 1$ then $P = B_n + 1$ .Let V be the set of vertices in G and $V_1 \subset V$ which are non end vertices in G. Again consider a subset $V_2 \subset V_1$ which are also non end vertices of G. If $V_1 - V_2 = D$ has no isolated vertex. Then D is a total dominating set, which gives $\gamma_t(G) = |D|$. Hence $|b_i| \leq P - |D|$ which gives $\gamma_{bct}\ (G) \leq P - \gamma_t(G)$.

$case$2: Suppose G is not a tree. Then there exists at least one block which is not an edge. Let $B_1, B_2, B_3, \dots \dots \dots B_n$ be the blocks of G and $b_1, b_2, b_3, \dots \dots \dots b_n$ be the corresponding block vertices in $B(G)$. Since each block of $B(G)$ is a complete and if a vertex $v \in D$ there exists a vertex $u \in V[B(G)] - D$ such that $N(u) \cap D = \{v\}$ is a minimal cototal dominating set D of $B(G)$. Then $|D|$ gives cototal block domination number $|D| = \gamma_{bct}(G) in\ B(G)$. Let $V(G)$ be the set of vertices of G .Let $D_1 \subset V(G)$ such that $V(G) - D_1$ gives a disconnected graph and every vertex of $V(G) - D_1$ is adjacent to at least one vertex of D_1.Then D_1 is a total dominating set. Hence $|D_1| = \gamma_t(G)$ is the minimum total dominating set. Clearly $|D| \leq P - |D_1|$ which gives $\gamma_{bct}\ (G) \leq P - \gamma_t(G)$.

Theorem 8 : For any graph $G\ , B(G) \neq K_2\ and\ K\ _{1,n}\ , n \geq 3\ then\ \gamma_{bct}\ (G) \leq \gamma_t(G) + \beta_O(G) - 3$

$Proof$: From the definition of co total domination $B(G) \neq K_2\ and\ \ K\ _{1,n}\ ,\ \ n \geq 3$ .Suppose $S = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the blocks of G .Then $M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the corresponding block vertices in $B(G)$ with respect to the set S . Let $H = \{v_1, v_2, v_3, \dots \dots \dots v_n\}$be the set of vertices in $G, V(G) = H$. we have the following cases.

case 1: Suppose G is acyclic. Let $H_1 = \{v_1, v_2, v_3, \dots \dots \dots v_i\}$,$1 \le i \le n$, $H_1 \subset H$ such that $\forall\, v_i \in H_1$ is an end vertex in G.Then. $H_2 \subseteq H$,where $\forall\, v_j \in H_2$ are at a distance at least two from each vertex $v_i \in H_1$. Then $|H_1 \cup H_2| = \beta_O$.

Let $J_1 = \{v_1, v_2, v_3, \dots \dots \dots v_i\}$,$1 \le i \le n$ are non end vertices in$G$. Suppose ${J_1}^1 \subset J_1$,$\forall\, v_j \in {J_1}^1$ are adjacent to atleast one vertex of J_1. The induced sub graph $D = \langle J_1 - {J_1}^1 \rangle$ has no isolated vertex which is minimal. Then $|D| = \gamma_t(G)$.

Let D_1 be a block cototal dominating set in $B(G)$. If a vertex $v \in D_1$ then there exists a vertex $u \in V[B(G)] - D_1$ such that $N(U) \cap D_1 = \{v\}$ is an isolated vertex which gives a minimal cototal dominating set. Clearly $|D_1| = \gamma_{bct}$ which gives $|D_1| \le |D| + |H_1 \cup H_2| - 3$.

Hence $\gamma_{bct}(G) \le \gamma_t(G) + \beta_O(G) - 3$.

case 2:Suppose G is cyclic there exists atleast one block which is cyclic or contains a cycle in G. Let $H_1 = \{v_1, v_2, v_3, \dots \dots \dots v_i\}$,$1 \le i \le n$, $H_1 \subset H$ and $H_2 = \{v_1, v_2, v_3, \dots \dots \dots v_s\}$,$1 \le s \le n$, $H_2 \subset H. Since\ H_1 \cap H_2 = \emptyset$,then for every vertex in $H_1\ and\ H_2$ which are incident to exactly one vertex in H. Therefore $H_1 \cup H_2$ is a independent set in G which gives $|H_1 \cup H_2| = \beta_O(G)$.

Let $U \subset V(G) = H$, $\forall\, v_i \in U$ is a cut vertex in G and $U_1 \subset H$ such that $\forall\, v_i \in U_1$ which are adjacent to atleast one vertex in U such that $U \cap U_1 = \emptyset$. Then $\langle D_2 \rangle = U \cap U_1$ exists which have no isolated vertex, defines total dominating set which gives $|D_2|$ as minimum total dominating set.Clearly $|D_2| = \gamma_t(G)$

Let $M_1 \subseteq M$,$\forall\, v_s \in M_1$ is an end vertex in $B(G)$, also $M_2 \subseteq M\ \forall v_j \in M_2$ are cut vertices which are adjacent to atleast one vertex in $v_s \in M_1$ such that $M_1 \cup M_2$ defines co total dominating set and gives $|M_1 \cup M_2| = |D_3| = \gamma_{bct}(G)$.

Hence $|D_3| \le |D_2| + |H_1 \cup H_2| - 3\ \ which\ \ implies\ \gamma_{bct}(G) \le \gamma_t(G) + \beta_O(G) - 3$.

Theorem 9: For any graph G , $B(G) \ne K_2\ and\ K_{1,n}$, $n \ge 3$ then $\gamma_{bct}(G) \le \gamma_c(G)$.

$Proof$: For cototal domination, we consider the graphs with the property $B(G) \neq K_2$ and $K_{1,n}, n \geq 3$.

We consider the following cases

$case 1$: Suppose each block is an edge in G.Let $V(G) = \{v_1, v_2, v_3, \dots \dots \dots v_n\}$ and $V_1(G) = \{v_1, v_2, v_3, \dots \dots \dots v_i\}$ where $1 \leq i \leq n$ $where$ $V_1(G) \subset V(G)$ for every v_i is an end vertex in$G$. The minimal connected dominating set is given by $\langle V(G) - V_1(G) \rangle$.Hence $|V(G) - V_1(G)| = \gamma_c(G)$.

Let $M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the block vertices in $B(G)$ corresponding the blocks

$S = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ since each block of G gives end vertices in $B(G).Then\ M_1 = \{b_i\}, 1 \leq i \leq n, M_1 \subset M\ in$ which every b_i is an end vertex. Suppose ${M_1}^1 = \{b_j\}$, $1 \leq j \leq n$, ${M_1}^1 \subset M$ every b_j is a cut vertex in $B(G)$.Let $M_2 \subset {M_1}^1$ and $\langle M - \{M_2 \cup M_1\} \rangle$ has no isolated vertex. Then $|M_2 \cup M_1| = \gamma_{bct}(G)$. Clearly $|M_2 \cup M_1| \leq |V(G) - V_1(G)|$ $which$

$gives\ \gamma_{bct}(G) \leq \gamma_c(G)$.

$case\ 2$: Suppose there exists at least one block which is not an edge .Let $K = \{B_1, B_2, B_3, \dots \dots \dots B_i\}$ be the sub set of blocks,$\forall B_i \in K$ has at least three vertices. Then the cardinality of S will increase. But in case of $B(G)$ each block becomes a vertex in $B(G)$. $Let\ M_1$ be the minimal dominating set of $B(G)$, such that $\langle M - M_1 \rangle$ has no isolates. Hence

$|M_1| = \gamma_{bct}(G)$. Since $V[B(G)] \subset V(G)$. We consider a set $D = \{v_1, v_2, v_3, \dots \dots \dots v_n\} \subset V(G)$ such that $\langle D \rangle$ is connected with minimal cardinality. Hence $|D| = \gamma_c(G)$. Clearl $|M_1| \leq |D|$ $which\ gives$ $\gamma_{bct}(G) \leq \gamma_c(G)$.

Theorem 10 : For any graph $G, B(G) \neq K_2$ and $K_{1,n}, n \geq 3$ $then$ $\gamma_{bct}(G) \leq \gamma_{ns}(G)$

$Proof$: For block cototal domination, we consider the graphs with the property such that $B(G) \neq K_2 and\ K_{1,n}\ n \geq 3$. Let $V(G) = \{v_1, v_2, v_3, \dots \dots \dots v_n\}$ and D is a dominating set of $G. If\ a\ vertex\ v \in D$ there exists a vertex $u \in V(G) - D$ such that $N(U) \cap D = \{v\}$ gives minimum non split dominating set such that $|D| = \gamma_{ns}(G)$ we have the following cases.

$case1$: Suppose each block of G is an edge. Then in $B(G)$ each block is complete. Let $M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be a set of vertices in $B(G)$ which corresponds to the blocks $B_1, B_2, B_3, \dots \dots \dots B_n$ of $G. Let\ M_1 = \{b_1, b_2, b_3, \dots \dots \dots b_i\}, 1 \le i \le n, M_1 \subset M$ be a dominating set in $B(G)$ which are adjacent to atleast one vertex in $V[B(G)] - M_1$. Then

$\langle M - M_1 \rangle$ has no isolated vertex, gives cototal domination set. Clearly $|M - M_1| = \gamma_{bct}(G)$. Hence $|M - M_1| \le |D|\ which\ gives\ \gamma_{bct}(G) \le \gamma_{ns}(G)$.

$case2$: Suppose each block of G is not an edge.Then $G\ is$ not a tree. Hence each block contains at least three vertices in $G. Let\ M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$ be the block vertices in $(G). Suppose\ J_1 = \{b_1, b_2, b_3, \dots \dots \dots b_i\}$, $1 \le i \le n,\ J_1 \subset M$ which are end vertices in $B(G)$. Let $J_2 = \{b_j\}, 1 \le b_j \le n, J_2 \subset M$. Every b_j is a cut vertex in $B(G)$. Suppose

$J_3 = \{b_s\}, 1 \le s \le n,\ J_3 \subset J_2$. Clearly $\langle J_3 \cup J_1 \rangle$ is a cototal dominating set. Then $|J_3 \cup J_1| = \gamma_{bct}(G)$. since at least one block of G contains atleast three vertices.Then cardinality of $\gamma_{ns} - set\ will\ increase$.Hence one can easily verify that $|J_3 \cup J_1| \le |D|$ which gives

$\gamma_{bct}(G) \le \gamma_{ns}(G)$.

Theorem 11: For any graph G , $B(G) \ne K_2$ and $K_{1,n}$,$n \ge 3$ then $\gamma_{bct}(G) + \gamma_{cot}(G) \le P$.

$Proof$: Suppose $B(G)$ is a complete graph, by definition cototal domination $B(G) \ne K_2\ or$

$K_{1,n}\ \ n \ge 3$.

Let $S = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the set of blocks of G. Then $M = \{b_1, b_2, b_3, \dots \dots \dots b_n\}$

be the corresponding block vertices in $B(G)$. Let $M_1 = \{b_1, b_2, b_3, \dots \dots \dots b_i\}, 1 \le i \le n$,

$M_1 \subset M$ are the end vertices in $B(G)$. Let $M_2 = \{b_1, b_2, b_3, \dots \dots \dots b_j\}, 1 \le j \le n$, $M_2 \subset M$ which are non end vertices in $B(G)$. Again $M_3 = \{b_1, b_2, b_3, \dots \dots \dots b_s\}, 1 \le s \le j$ such that $M_3 \subset M_2$. Then $\langle M - \{M_2 \cup M_3\} \rangle$ has no isolates. Hence $|M_2 \cup M_3| = \gamma_{bct}(G)$.

Let $V(G) = \{v_1, v_2, v_3, \dots \dots \dots v_p\}, H = \{v_1, v_2, v_3, \dots \dots \dots v_i\}, 1 \le i \le p$ be a subset of $V(G)$ which are end vertices in G. Let $J = \{v_1, v_2, v_3, \dots \dots \dots v_j\} \subseteq V(G)$ with $1 \le j \le p$

$such\ that\ \forall\ v_j \in J\ , N(v_i) \cap N(v_j) = \emptyset$, then $\langle V(G) - \{H \cup J\}\rangle$ has no isolates. Thus

$|H \cup J| = \gamma_{cot}(G)$. Now $|M_2 \cup M_3| + |H \cup J| \leq |V(G)|$,which gives$\gamma_{bct}(G) + \gamma_{cot}(G) \leq P$.

Theorem 12 : If v be an end vertex of $B(G)$, then v is in every $\gamma_{bct} - set$. If $B(G) \neq K_2\ , and\ K_{1,n}\ , n \geq 3$.

Proof :For cototal domination, we consider the graphs with the property such that $B(G) \neq K_2\ , and\ K_{1,n}\ , n \geq 3$.

Let $D = \{v_1, v_2, v_3 \ldots \ldots \ldots v_n\} \subseteq V[B(G)]$ be the minimal cototal block dominating set of G. suppose there exists a vertex set $D^{-1} \subseteq V[B(G)] - D$ be the $\gamma_{bct} - set\ of\ G$. Assume there exists an end vertex $v \in V[B(G)]\ , v \in D^{-1}$. Now consider any two vertices $u\ and\ w\ such\ that$

$u\ , w\ \notin D^{-1}$. Since $v \in D^{-1}$,$v\ is\ in\ every\ u - w\ path\ in\ B(G)$.Further, since $\deg(v) = 1$

$where\ v \in V[B(G)]$ it follows that the set $D^1 = (\ D^{-1} - \{u, w\}) \cup \{v\}$ is also a minimal cototal dominating set of $B(G)$. Clearly $|D^1| = |D^{-1}| = 1$, a contradiction to the fact that D^{-1} is also a $\gamma_{bct} - set\ of\ G. Hence\ \ u \in D^{-1}$ and $v\ is$ in every $\gamma_{bct} - set\ of\ G$.

Theorem 13: For any connected graph G with $n - blocks\ \ \overline{B(G)} \neq K_2\ or\ K_{1,n}\ and\ n \geq 3$ then $\gamma_{ct}\left[\overline{B(G)}\right] \leq n - 2$

Proof : From the definition of co total domination $\overline{B(G)}\ \neq K_2\ or\ K_{1,n}\ and\ n \geq 3$. Suppose $S = \{B_1\ , B_2\ , B_3\ , \ldots \ldots \ldots B_n\ \}$ be the blocks of G . Then $M = \{b_1, b_2, b_3, \ldots \ldots \ldots b_n\}$ be the corresponding block vertices in $B(G)\ and\ \overline{B(G)}$ with respect to the set S . Let $M_1 = \{b_1, b_2, b_3, \ldots \ldots \ldots b_i\}\ , 1 \leq i \leq n\ , M_1 \subset M$ for all $b_i \in M_1$ which are end vertices in $\overline{B(G)}$. Again

$M_2 = \{b_1, b_2, b_3, \ldots \ldots \ldots b_s\}\ , 1 \leq s \leq n\ , M_2 \subset M$ which are non end vertices in $\overline{B(G)}$. Also $M_3 = \{b_1, b_2, b_3, \ldots \ldots \ldots b_j\}$,$1 \leq j \leq s\ , M_3 \subset\ M_2$ such that $\forall\ b_j \in M_3$ which are also non end vertices in $\overline{B(G)}$ which are adjacent to atleast one non end vertex in $\overline{B(G)}$. The induced sub graph $\langle M - (M_1 \cup M_3)\rangle$ has no isolated vertices. Then $|M_1 \cup M_3| = \gamma_{ct}\left[\overline{B(G)}\right]$. Suppose

$M_1 = \emptyset$ then $|M_1 \cup \emptyset|$ has no isolated vertices which gives minimum co total domination. Clearly $\gamma_{ct}\left[\overline{B(G)}\right] \leq n-2$.

Further we developed the following theorem of Nordhaus- Gaddum type- Results.

Theorem 14: If $G \; and \; \overline{G}$ are connected graph, $B(G) \; and \; \overline{B(G)} \neq K_2 \; or \; K_{1,n} \; and \; n \geq 3$ then $i)$ $\gamma_{ct}[B(G)] + \gamma_{ct}[\overline{B(G)}] \leq 2\,(n-2)$

$ii)$ $\gamma_{ct}[B(G)] \,.\, \gamma_{ct}[\overline{B(G)}] \leq (n-2)^2$

$Proof$: From $Theorem$ (1) and $Theorem$ (13) the above results follows.

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