A NUMERICAL SOLUTION OF A BURGER'S EQUATION BY USE OF LINE METHOD

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Abstract

The Burgers equation models very well fluid flow nature problems.Its a very important mathematical model used both in Engineering and Applied Mathematics.The theory behind the Burgers equation is extremely rich and interesting .This paper is concerned with Lines descritization approach that eventually transforms the model partial differential equations into a system of first order ordinary differential equations by use of different initial and boundary conditions from those used by other authors and comparison is made based on solutions obtained

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1. Introduction

Burgers' equation was proposed as a mathematical model of turbulence [1]. Since then, this model equation has been found applicable in many disciplines such as number theory, gas dynamics, heat conduction, elasticity etc [3]. Since Burgers' equation involves non linear advection term and dissipation term, it is used to simulate wave motion [6]. Comparison of numerical solutions can be made due to having analytical solutions of the Burgers' equation. Various numerical techniques have been applied to solve the equation. The Burgers' equation is a non linear partial differential equation of second order as follows

$$\frac{\partial w}{\partial t}(t,x)+w(t,x)\frac{\partial w}{\partial x}(t,x)=\varepsilon\frac{\partial^2 w}{\partial x^2}(t,x)+f(t,x) \qquad (1)$$

Where $\frac{\partial w}{\partial t}(t,x)$ is the dependent variable

$w(t,x)\frac{\partial w}{\partial x}(t,x)$ is the convective term.

$\frac{\partial^2 w}{\partial x^2}(t,x)$ is the diffusive term

$f(t,x)$ is the forcing term

ε Is the viscosity constant.

It is used in fluid dynamics teaching and in engineering as a simplified model for turbulence, boundary layer behaviour, shock wave formation and mass transport. It has been studied and applied for many decades. Many different closed form, series approximations and numerical solutions are known for particular set of boundary conditions. The study of the properties of Burgers' equation has attracted considerable attention due to its application in various theories. The equation plays a very vital role both for the conceptual understanding of a class of physical flows and for testing various numerical algorithms. A great deal of effort has been expended in the past years to compute efficiently the numerical solutions of this great equation. The Burgers' equation has been used to study a number of physically important phenomena, including shock waves, acoustic transmission and traffic flow [1]. Besides its importance in understanding convection-diffusion phenomena, Burgers' equation can be used, especially for computational purposes for fluid flow problems [4].

In spite the fact that the numerical solution of Burgers' equation has received a fair amount of attention it seems to draw more and more attention every day. Actually some of the importance of Burgers' equation stems from the fact that it is one of the few non-linear equations with known exact solutions in low dimensions.

However the equation gets its name from the extensive research of Burgers' [2] beginning in 1939. He basically focused on modeling turbulence, but the equation is useful in modeling such diverse physical phenomena as shock flows, traffic flow, acoustic transmission in fog etc. In fact, it can be used as a model for any non-linear wave propagation problem subject to dissipation.

Depending on what one is modeling this dissipation may result from viscosity, heat conduction, mass diffusion, thermal radiation, chemical reaction, or other source. We intend to take the point of view that Burgers' equation is a perturbation of the linear heat equation. In particular, we consider the Burgers' Equation without a forcing term. Let's consider it in the form

$$u_t + uu_x - \mu u_{xx} = 0, \quad 0 \le x \le 1, \quad 0 \le t \le 0.5 \qquad (2)$$

2. Method of Lines

Now there will be need to descritize the partial derivatives to come up with ordinary differential equations. Consider the uniform mesh $h = \frac{b-a}{N}$ for $i = 0,1,2.............N$. Let's have a size $h = 0.1$ And we apply the centered differences for

$$u_x = \frac{u_{i+1} - u_{i-1}}{2h} \qquad (3)$$

$$u_{xx} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \qquad (4)$$

Now substituting (3) and (4) into (2) we come up with the below equations

$$\frac{du_i}{dt} + u_i\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) - \mu\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) = 0 \qquad (5)$$

Making u_{i+1} then our equation (5) above becomes

$$u_{i+1} = -\frac{2h^2}{hu_i - 2\mu} u_{t\ i} - \frac{4\mu}{hu_i - 2\mu} u_i + \frac{hu_i + 2\mu}{hu_i - 2\mu} u_{i-1} \qquad (6)$$

In matrix form (6) can be written as

$$\begin{bmatrix} u_2 \\ u_3 \\ . \\ . \\ . \\ u_{10} \end{bmatrix} = \begin{bmatrix} -\dfrac{2h^2}{hu_1 - 2\mu} u_{t\ 1} \\ -\dfrac{2h^2}{hu_2 - 2\mu} u_{t\ 2} \\ . \\ . \\ . \\ -\dfrac{2h^2}{hu_9 - 2\mu} u_{t\ 9} \end{bmatrix} - \begin{bmatrix} \dfrac{4\mu}{hu_1 - 2\mu} u_1 \\ \dfrac{4\mu}{hu_2 - 2\mu} u_2 \\ . \\ . \\ . \\ \dfrac{4\mu}{hu_9 - 2\mu} u_9 \end{bmatrix} + \begin{bmatrix} \dfrac{hu_1 + 2\mu}{hu_1 - 2\mu} u_0 \\ \dfrac{hu_2 + 2\mu}{hu_2 - 2\mu} u_1 \\ . \\ . \\ . \\ \dfrac{hu_9 + 2\mu}{hu_9 - 2\mu} u_8 \end{bmatrix} \quad (7)$$

Clearly (7) forms a system of ordinary differential equations

3. Results

Let's consider the following initial and boundary conditions and compare with results obtained by [4].

$$u(0,t) = \frac{1}{4}\cos(\pi x + t\mu)\ ,\ \frac{\partial u}{\partial x}\ 0,t\ = -\frac{e^{\cos \pi\mu t}}{\left(1 + e^{\cos\frac{\pi}{\mu}}\right)} \quad (8)$$

With exact solutions

$$u\ \ x,t\ = \frac{1}{1 + e^{\left(\frac{x}{2\mu} - \frac{t}{4\mu}\right)}} \quad (9)$$

The solutions obtained are put in the table below

Table1. Solution of $u(x,t)$

x_i	Numerical Values t =0	Error t =0	Numerical Values t =0.25	Error t =0.25	Numerical Values t =0.5	Error t =0.5
0.0	0.500000	0.00000	0.515619	0.00000	0.531208	0.000000
0.1	0.487500	0.000002	0.503131	0.000006	0.518848	0.000016
0.2	0.475016	0.000003	0.490640	0.000012	0.506384	0.000033
0.3	0.462563	0.000004	0.478261	0.000024	0.493805	0.000053
0.4	0.450211	0.000004	0.465711	0.000035	0.481436	0.000075
0.5	0.437855	0.000005	0.453307	0.000046	0.468992	0.0000101
0.6	0.425400	0.000006	0.440801	0.000057	0.456433	0.0000134
0.7	0.412911	0.000004	0.428403	0.000067	0.444012	0.0000171
0.8	0.400401	0.000005	0.416111	0.000078	0.431753	0.0000210
0.9	0.388109	0.000004	0.403800	0.000088	0.419311	0.0000250
1.0	0.375711	0.000003	0.391545	0.000099	0.407022	0.0000293

4. Conclusion

In this paper we obtain an effective and also efficient method for solving Burgers' equations which converges much faster than other methods used by other authors and the errors involved are also minimal or rather manageable. Change of initial and boundary conditions vary the solutions slightly but retain a steady solution format.

References

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