COMMON FIXED POINT THEOREMS FOR TWO MAPS IN CONE HEXAGONAL METRIC SPACE

P.R.Sobhana babu[*]

ABSTRACT:

The existence of unique common fixed point theorems for two weakly compatible self-maps satisfying different contractive conditions in cone hexagonal metric space

Keywords: weakly compatible maps, common fixed points, cone hexagonal metric space, coincidence points

[*] Department of mathematics, Ramachandra College of engineering, Eluru-534007, A.P, India

1. Introduction:

The fundamental work in metric fixed point theory, by Stefan banach in 1922 is famous as banach contraction principle. After that a number of authors introduced different contractive type mappings and proved many fixed point theorems extending the theory. B.E.Rhoades [1], Paula Collaco and Jaime Carvalho E Silva [2] compared various definitions of contractive mappings. In 2007, Huang and Zhang [3] introduced the concept of cone metric spaces by replacing the co domain with Banach spaces in a metric function whose range satisfy the properties of a cone. Subsequently, Abbas and Jungck [4], Abbas and Rhoades [5] have studied common fixed point theorems in cone metric spaces for normal cones with the assumption of normality. Sh.Rezapour and R.Hamlbarani [6] proved some fixed point theorems for any cone, omitting the assumption of normality. Recently, several authors have proved and proving many common fixed point theorems, see [7-12]. Recently Manoj Garg [14] introduced cone hexagonal metric space and proved banach contraction principle in cone hexagonal metric space.

The purpose of this paper is to prove the common fixed point theorems for a pair of weakly compatible maps which satisfy generalized contractive conditions in complete normal cone hexagonal metric space.

The following notions have been used to prove the main result.

Definition 1.1:([3]) Let E be a real Banach Space. A subset P of E is called cone if and only if

(i) P is closed, non empty and $P \neq \{0\}$ (ii) $0 \leq a, b \in R$ and $x, y \in P$ implies $ax + by \in P$

(iii) $P \cap (-P) = \{0\}$.

Definition 1.2:([3]) The partial ordering $\leq$ with respect to $P \subseteq E$ is defined by (i) $x \leq y$ if and only if $y - x \in P$. (ii) $x < y$ shows that $x \leq y$ but $x \neq y$, (iii) $x << y$ will stand for $y - x \in int(P)$, int(P) denotes the interior of P.

Definition 1.3 :([3]) A cone P is called normal if there is a number $\mu \geq 1$ such that for all $x, y \in E$, the inequality $0 \leq x \leq y$ implies $\|x\| \leq \mu \|y\|$.

The least positive number μ satisfying the above inequality is called the normal constant of P.

In this paper we always suppose that E is a real Banach space and P is a cone in E with int(P) $\neq \Phi$ and

'$\leq$' is a partial ordering with respect to P.

Definition 1.4:([13]) Let X be a non empty set. Suppose that the mapping d: X × X → E satisfies:

(i) 0 < d(x, y) for all x, y ϵ X and d(x, y) = 0 if and only if x = y.

(ii) d (x, y) =d (y, x) for all x, y ϵ X.

(iii) d (x, y) ≤ d(x, z)+d(z, w)+d(w, u) + d(u, y) for all x, y, z, w, u ϵ X and for all distinct points z, w, u ϵ X - {x, y}.

Then d is called a cone pentagonal metric on X and (X,d) is called a cone pentagonal metric space.

Definition 1.5:[14] Let X be a non empty set. Suppose that the mapping d : X × X → E satisfies:

(i) 0 < d(x, y) for all x, y ϵ X and d(x, y) = 0 if and only if x = y

(ii) d (x, y) =d (y, x) for all x, y ϵ X.

(iii) d(x, y) ≤ d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, y) for all x, y, z, w, u, v ϵ X and for all distinct points z, w, u, v ϵ X -{x, y}[hexagonal property].

Then d is called a cone hexagonal metric on X, and (X, d) is called a cone hexagonal metric space.

Definition 1.6[14]: Let $\{x_n\}$ be a sequence in a cone hexagonal metric space (X, d) and xϵ X. If for every c ϵ E, with 0<< c there exist n_0 ϵ N and that for all $n > n_0$, $d(x_n, x) << c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, as $n\to\infty$.

Definition 1.7[14]: If for every c ϵ E, with 0 << c there exist n_0 ϵ N such that for all $n > n_0$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called Cauchy sequence in a cone hexagonal metric space X.

Definition 1.8[14]: If every Cauchy sequence is convergent in a cone hexagonal metric space (X, d), then (X, d) is called a complete cone hexagonal metric space.

Theorem 1.9[14]: Let (X, d) be a cone hexagonal metric space and P be a normal cone with normal constant μ. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $\|d(x_n, x)\| \to 0$ as $n\to\infty$

Theorem 1.10[14]: Let (X, d) be a cone hexagonal metric space and P be a normal cone with normal constant μ. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $\|d(x_n, x_{n+p})\| \to 0$ as $n \to \infty$.

Theorem 1.11[14]: Every cone (or rectangular or pentagonal) metric space is cone hexagonal metric space.

The converse of the above theorem is not necessarily true as it can be seen from the following example.

Example 1.12 Let X = N, E = R^2 and P= {(x, y): x, y ≥ 0}. Define d: X × X → E as follows:

d(x, y) = (0, 0) if x = y;

d(x, y) = (9, 15) if x and y are in {3, 4}, x ≠ y;

d(x, y) = (3, 5) if x and y cannot both at a time in {3, 4}, x ≠ y.

Then (X, d) is a cone hexagonal (or pentagonal or rectangular) metric space but not a cone metric space because it lacks the triangular property:

(9, 15) = d (3, 4) > d (3, 5) + d (5, 4) = (3, 5) + (3, 5) = (6, 10)

As (9, 15) - (6, 10) = (3, 5) ϵ P.

Theorem 1.13[14]: Every rectangular and pentagonal (resp. complete rectangular and complete pentagonal) cone metric space is hexagonal (resp. complete hexagonal) cone metric space.

The converse of the above theorem is not necessarily true as it can be seen from the following example.

Example 1.14: let $X = \{1,2,3,4,5,6\}, E = R^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a normal cone in E. Define

d: X × X → E as follows:

$d(1,2) = d(2,1) = (5,10)$

$d(1,3) = d(3,1) = d(1,4) = d(4,1) = d(1,5) = d(5,1) = d(2,3) = d(3,2) = d(2,4) = d(4,2) = d(2,5) = d(5,2)$
$= d(3,4) = d(4,3) = d(3,5) = d(5,3) = d(4,5) = d(5,4) = (1,2)$

$d(1,6) = d(6,1) = d(2,6) = d(6,2) = d(3,6) = d(6,3) = d(4,6) = d(6,4) = d(5,6) = d(6,5) = (4,8)$

Then (X, d) is a cone hexagonal (resp. complete hexagonal) metric space but not a cone pentagonal (resp. complete pentagonal) and so cone rectangular (resp. complete rectangular) metric space because it lacks the pentagonal and rectangular property:

(5, 10) = d(1, 2) > d(1, 3) + d(3, 4) + d(4, 5) +d(5, 2) = (1, 2) + (1, 2) + (1, 2) + (1, 2) = (4, 8)

As (5, 10) - (4, 8) = (1, 2) ϵ P.

Definition 1.15 [4]: let X be a non empty set and f, g two self maps on X then

(i) if q=fp=gp for some p ϵ X, then p is called a coincidence point of f and g also q is called a point of coincidence of f and g

(ii) if p=fp=gp for some p ϵ X, then p is called a common fixed point of f and g

Definition 1.16[4]: let X be a non empty set and f, g two self maps on X . The pair {f, g} is

Said to be weakly compatible if f(gt) = g(ft) Whenever ft=gt for some t ϵ X

Theorem 1.17[4]: let f and g weakly compatible self maps of a set X. If f and g have a unique point of coincidence w=fx=gx, then w is the unique common fixed point of f and g.

2. Main results

Theorem 2.1: let (X, d) be a cone hexagonal metric space . Suppose the self maps $f, g: X \rightarrow X$ satisfy the contractive condition $d(fx, fy) \leq kd(gx, gy), \forall x, y \in X$

2.1(1)

Where $k \in [0,1)$ is a constant. If the range of f is contained in the range of g and the range of g is a complete subspace of X. more over if f and g are weakly compatible, then f and g have unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary choose $x_1 \in X$ such that $f(x_0) = g(x_1)$ .This is possible since $f(X) \subseteq g(X)$ continuing this process, choose $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$, n= 0, 1, 2…….

Now we define a sequence {y_n} in X such that $y_n = f(x_n) = g(x_{n+1})$ for n=0, 1, 2……

2.1(2)

If $y_m = y_{m+1}$ for *some* $m \in N$, then $y_m = f(x_{m+1}) = g(x_{m+1})$

That is f and g have a coincidence point $x_{m+1} \in X$

Assume $y_n \neq y_{n+1}, \forall n \in N$

Now $d(y_n, y_{n+1}) = d(fx_n, fx_{n+1})$

$$\leq kd(gx_n, gx_{n+1}) \quad \text{Using 2.1(1)}$$

$$\leq kd(y_{n-1}, y_n) \quad \text{Using 2.1(2)}$$

By repeated application of 2.1(1) and 2.1(2), we get

$$d(y_n, y_{n+1}) \leq k^n d(y_0, y_1)$$

2.1(3)

Now $d(y_n, y_{n+2}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})$

$$\leq k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1)$$

$$\leq k^n (1+k) d(y_0, y_1)$$

$$d(y_n, y_{n+2}) \leq \frac{k^n}{1-k} d(y_0, y_1)$$

2.1(4)

Similarly $d(y_n, y_{n+3}) \leq \frac{k^n}{1-k} d(y_0, y_1)$

2.1(5)

$$d(y_n, y_{n+4}) \leq \frac{k^n}{1-k} d(y_0, y_1)$$

2.1(6)

Now for the sequence $\{y_n\}$ we consider $d(y_n, y_{n+p})$ in four cases

Case I: p= 4m+1 for m ≥ 1

Then by hexagonal equality we have

$$d(y_n, y_{n+4m+1}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+4m+1})$$

$$\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5})$$
$$+ d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+7}) + d(y_{n+7}, y_{n+8}) + d(y_{n+8}, y_{n+4m+1})$$

$$= d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) +$$
$$\ldots\ldots\ldots\ldots\ldots\ldots + d(y_{n+4m-1}, y_{n+4m}) + d(y_{n+4m}, y_{n+4m+1})$$

$$\leq k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + +k^{n+2} d(y_0, y_1) + \ldots\ldots + k^{n+4m-1} d(y_0, y_1) + k^{n+4m} d(y_0, y_1)$$

$$\leq k^n(1+k+k^2+k^3+..........)d(y_0,y_1)$$

Hence $d(y_n, y_{n+4m+1}) \leq \dfrac{k^n}{1-k} d(y_0,y_1)$, **m ≥ 1** **2.1(7)**

Case II: p= 4m+2 for m ≥ 1

By hexagonal equality we have

$$d(y_n,y_{n+4m+2}) \leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+4m+2})$$

$$\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5}) + d(y_{n+5},y_{n+6})+d(y_{n+6},y_{n+7})+d(y_{n+7},y_{n+8})+d(y_{n+8},y_{n+4m+2})$$

$$= d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5})+ ....+d(y_{n+4m},y_{n+4m+1})+d(y_{n+4m+1},y_{n+4m+2})$$

$$\leq k^n d(y_0,y_1)+k^{n+1}d(y_0,y_1)++k^{n+2}d(y_0,y_1)+.......+k^{n+4m}d(y_0,y_1)+k^{n+4m+1}d(y_0,y_1)$$

$$\leq k^n(1+k+k^2+k^3+..........)d(y_0,y_1)$$

Hence $d(y_n, y_{n+4m+2}) \leq \dfrac{k^n}{1-k} d(y_0,y_1)$, **m ≥ 1** **2.1(8)**

Case III: p= 4m+3 for m ≥ 1

By hexagonal equality we have

$$d(y_n,y_{n+4m+3}) \leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+4m+3})$$

$$\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5}) + d(y_{n+5},y_{n+6})+d(y_{n+6},y_{n+7})+d(y_{n+7},y_{n+8})+d(y_{n+8},y_{n+4m+3})$$

$$= d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5})+ ....+d(y_{n+4m+1},y_{n+4m+2})+d(y_{n+4m+2},y_{n+4m+3})$$

$$\leq k^n d(y_0,y_1)+k^{n+1}d(y_0,y_1)++k^{n+2}d(y_0,y_1)+.......+k^{n+4m+1}d(y_0,y_1)+k^{n+4m+2}d(y_0,y_1)$$

$$\leq k^n(1+k+k^2+k^3+..........)d(y_0,y_1)$$

Hence $d(y_n, y_{n+4m+3}) \leq \dfrac{k^n}{1-k} d(y_0,y_1)$, **m ≥ 1** **2.1(9)**

Case IV: p= 4m+4 for m ≥ 1

By hexagonal equality we have

$$d(y_n,y_{n+4m+4}) \le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+4m+4})$$

$$\le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5})$$
$$+d(y_{n+5},y_{n+6})+d(y_{n+6},y_{n+7})+d(y_{n+7},y_{n+8})+d(y_{n+8},y_{n+4m+4})$$

$$= d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5})+$$
$$\dots\dots\dots\dots\dots\dots + d(y_{n+4m+2},y_{n+4m+3})+d(y_{n+4m+3},y_{n+4m+4})$$

$$\le k^n d(y_0,y_1)+k^{n+1}d(y_0,y_1)++k^{n+2}d(y_0,y_1)+\dots\dots+k^{n+4m+2}d(y_0,y_1)+k^{n+4m+3}d(y_0,y_1)$$

$$\le k^n(1+k+k^2+k^3+\dots\dots\dots..)d(y_0,y_1)$$

Hence $d(y_n,y_{n+4m+4}) \le \dfrac{k^n}{1-k}d(y_0,y_1)$, **m ≥ 1** **2.1(10)**

Thus from above four cases, we have $d(y_n,y_{n+p}) \le \dfrac{k^n}{1-k}d(y_0,y_1)$

Since P is normal cone we have $\|d(y_n,y_{n+p})\| \le \mu\dfrac{k^n}{1-k}\|d(y_0,y_1)\| \to 0$ as n→∞ for all p ϵ N

Hence $\{y_n\}$ is a Cauchy sequence in X.

Since $g(X)$ is a complete subspace of X, there exist b ϵ X such that

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_{n+1} = b$

Also we can find a ϵ X such that ga=b

Now consider

$$d(fa,b) \le d(fa,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},b)$$

$$= d(fa,fx_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},b)$$

$$\le kd(ga,gx_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},b)$$

$$\le kd(b,y_n)+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},b)$$

Since P is normal cone we have

$$\|d(fa,b)\| \le \mu\|kd(b,y_n)+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},b)\| \to 0 \text{ as } n\to\infty$$

This implies $\|d(fa,b)\| = 0$

Which implies $fa = b$ **2.1(11)**

Therefore $fa = ga = b$ thus f and g have a coincidence point in X.

Now we shall prove point of coincidence is unique.

Let r be any point of coincidence of f and g Then $fy = gy = r$ **2.1(12)**

Suppose v is any point of coincidence of f and g then we have $fc = gc = v$ **2.1(13)**

Now, $d(r,v) = d(fy, fc) \leq kd(gy, gc)$

$d(r,v) \leq kd(r,v)$

$(k-1)d(r,v) \in P$ Using def 1.2(i)

Multiplying with positive real number (1-k), we get $-d(r,v) \in P$

But, we have $d(r,v) \in P$. From the def of cone and cone metric we get $r = v$

Thus f and g have unique point of coincidence in X

Since f and g are weakly compatible then from 2.1(12) we have $fr = fgy = gfy = gr$

Therefore $fr = gr = w(say)$

This shows that w is another point of coincidence of f and g

Therefore by the uniqueness of point of coincidence we must have $w = r$

Hence, there exist unique point r ϵ X such that $fr = gr = r$

Thus r is a unique common fixed point of self mappings f and g

Hence f and g have a unique common fixed point

Theorem 2.2: let (X, d) be a cone hexagonal metric space. Suppose the self maps $f, g : X \to X$ satisfy the contractive condition $d(fx, fy) \leq k[d(fx, gx) + d(fy, gy)], \forall x, y \in X$ **2.2(1)**

Where $k \in [0, \frac{1}{2})$ is a constant. If the range of f is contained in the range of g and the range of g is a complete subspace of X then f and g have a unique coincidence point in X more over if f and g are weakly compatible, then f and g have unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary choose $x_1 \in X$ such that $f(x_0) = g(x_1)$.This is possible since $f(X) \subseteq g(X)$ continuing this process, choose $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$, n= 0, 1, 2…….

Now we define a sequence $\{y_n\}$ in X such that $y_n = f(x_n) = g(x_{n+1})$ for n=0, 1, 2…… **2.2 (2)**

If $y_m = y_{m+1}$ for $some\, m \in N$, then $y_m = f(x_{m+1}) = g(x_{m+1})$

That is f and g have a coincidence point $x_{m+1} \in X$

Assume $y_n \neq y_{n+1}, \forall n \in N$

Now $d(y_n, y_{n+1}) = d(fx_n, fx_{n+1})$

$\leq k[d(fx_n, gx_n) + d(fx_{n+1}, gx_{n+1})]$ From 2.2(1)

$= k[d(y_n, y_{n-1}) + d(y_{n+1}, y_n)]$ From 2.2(2)

$= k[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]$

Which implies that $d(y_n, y_{n+1}) \leq \dfrac{k}{1-k} d(y_{n-1}, y_n)$ for n= 0, 1, 2…….

Thus $d(y_n, y_{n+1}) \leq \rho d(y_{n-1}, y_n)$

$\leq \rho^2 d(y_{n-2}, y_{n-1})$

……

……

$$d(y_n, y_{n+1}) \leq \rho^n d(y_0, y_1) \qquad \textbf{2.2(3)}$$

For all $n \geq 0$ where $\rho = \dfrac{k}{1-k} < 1$

Now $d(y_n, y_{n+2}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})$

$\leq \rho^n d(y_0, y_1) + \rho^{n+1} d(y_0, y_1)$

$\leq \rho^n (1+\rho) d(y_0, y_1)$

$$d(y_n, y_{n+2}) \leq \frac{\rho^n}{1-\rho} d(y_0, y_1) \qquad \textbf{2.2(4)}$$

Similarly $d(y_n, y_{n+3}) \leq \dfrac{\rho^n}{1-\rho} d(y_0, y_1)$ **2.2(5)**

$$d(y_n, y_{n+4}) \le \frac{\rho^n}{1-\rho} d(y_0, y_1) \qquad \textbf{2.2(6)}$$

Now for the sequence $\{y_n\}$ we consider $d(y_n, y_{n+p})$ in four cases

Case I: p= 4m+1 for m ≥ 1

Then by hexagonal equality we have

$$d(y_n, y_{n+4m+1}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+4m+1})$$

$$\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+7}) + d(y_{n+7}, y_{n+8}) + d(y_{n+8}, y_{n+4m+1})$$

$$= d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + \dots\dots + d(y_{n+4m-1}, y_{n+4m}) + d(y_{n+4m}, y_{n+4m+1})$$

$$\le \rho^n d(y_0, y_1) + \rho^{n+1} d(y_0, y_1) + +\rho^{n+2} d(y_0, y_1) + \dots\dots + \rho^{n+4m-1} d(y_0, y_1) + \rho^{n+4m} d(y_0, y_1)$$

$$\le \rho^n (1 + \rho + \rho^2 + \rho^3 + \dots\dots) d(y_0, y_1)$$

Hence $d(y_n, y_{n+4m+1}) \le \frac{\rho^n}{1-\rho} d(y_0, y_1)$, **m ≥ 1** **2.2(7)**

Case II: p= 4m+2 for m ≥ 1

By hexagonal equality we have

$$d(y_n, y_{n+4m+2}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+4m+2})$$

$$\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+7}) + d(y_{n+7}, y_{n+8}) + d(y_{n+8}, y_{n+4m+2})$$

$$= d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + \dots\dots + d(y_{n+4m}, y_{n+4m+1}) + d(y_{n+4m+1}, y_{n+4m+2})$$

$$\le \rho^n d(y_0, y_1) + \rho^{n+1} d(y_0, y_1) + +\rho^{n+2} d(y_0, y_1) + \dots\dots + \rho^{n+4m} d(y_0, y_1) + \rho^{n+4m+1} d(y_0, y_1)$$

$$\le \rho^n (1 + \rho + \rho^2 + \rho^3 + \dots\dots) d(y_0, y_1)$$

Hence $d(y_n, y_{n+4m+2}) \le \frac{\rho^n}{1-\rho} d(y_0, y_1)$, **m ≥ 1** **2.2(8)**

Case III: p= 4m+3 for m ≥ 1

By hexagonal equality we have

$$d(y_n, y_{n+4m+3}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+4m+3})$$

$$\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+7}) + d(y_{n+7}, y_{n+8}) + d(y_{n+8}, y_{n+4m+3})$$

$$= d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + \dots\dots\dots + d(y_{n+4m+1}, y_{n+4m+2}) + d(y_{n+4m+2}, y_{n+4m+3})$$

$$\le \rho^n d(y_0, y_1) + \rho^{n+1} d(y_0, y_1) + \rho^{n+2} d(y_0, y_1) + \dots\dots + \rho^{n+4m+1} d(y_0, y_1) + \rho^{n+4m+2} d(y_0, y_1)$$

$$\le \rho^n (1 + \rho + \rho^2 + \rho^3 + \dots\dots\dots) d(y_0, y_1)$$

Hence $d(y_n, y_{n+4m+3}) \le \dfrac{\rho^n}{1-\rho} d(y_0, y_1)$, **m ≥ 1** **2.2(9)**

Case IV: p= 4m+4 for m ≥ 1

By hexagonal equality we have

$$d(y_n, y_{n+4m+4}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+4m+4})$$

$$\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+7}) + d(y_{n+7}, y_{n+8}) + d(y_{n+8}, y_{n+4m+4})$$

$$= d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + \dots\dots\dots + d(y_{n+4m+2}, y_{n+4m+3}) + d(y_{n+4m+3}, y_{n+4m+4})$$

$$\le \rho^n d(y_0, y_1) + \rho^{n+1} d(y_0, y_1) + + \rho^{n+2} d(y_0, y_1) + \dots\dots + \rho^{n+4m+2} d(y_0, y_1) + \rho^{n+4m+3} d(y_0, y_1)$$

$$\le \rho^n (1 + \rho + \rho^2 + \rho^3 + \dots\dots\dots) d(y_0, y_1)$$

Hence $d(y_n, y_{n+4m+4}) \le \dfrac{\rho^n}{1-\rho} d(y_0, y_1)$, **m ≥ 1** **2.2(10)**

Thus from above four cases, we have $d(y_n, y_{n+p}) \le \dfrac{\rho^n}{1-\rho} d(y_0, y_1)$

Since P is normal cone we have $\|d(y_n, y_{n+p})\| \le \mu \dfrac{\rho^n}{1-\rho} \|d(y_0, y_1)\| \to 0$ as n→∞ for all p ϵ N

Hence $\{ y_n \}$ is a Cauchy sequence in X.

Since $g(X)$ is a complete subspace of X, there exist b ϵ X such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} g x_{n+1} = b$

Also we can find a ϵ X such that ga=b

Now consider

$$d(fa,b) \le d(fa,y_{n+1}) + d(y_{n+1},y_{n+2}) + d(y_{n+2},y_{n+3}) + d(y_{n+3},y_{n+4}) + d(y_{n+4},b)$$

$$= d(fa,fx_{n+1}) + d(y_{n+1},y_{n+2}) + d(y_{n+2},y_{n+3}) + d(y_{n+3},y_{n+4}) + d(y_{n+4},b)$$

$$\le k[d(fa,ga) + d(fx_{n+1},gx_{n+1})] + d(y_{n+1},y_{n+2}) + d(y_{n+2},y_{n+3}) + d(y_{n+3},y_{n+4}) + d(y_{n+4},b)$$

$$\le kd(fa,b) + kd(y_{n+1},y_n) + d(y_{n+1},y_{n+2}) + d(y_{n+2},y_{n+3}) + d(y_{n+3},y_{n+4}) + d(y_{n+4},b)$$

$$d(fa,b) \le \frac{1}{1-k}[kd(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + d(y_{n+2},y_{n+3}) + d(y_{n+3},y_{n+4}) + d(y_{n+4},b)]$$

Since P is normal cone we have

$$\|d(fa,b)\| \le \frac{\mu}{1-k}\|kd(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + d(y_{n+2},y_{n+3}) + d(y_{n+3},y_{n+4}) + d(y_{n+4},b)\| \to 0 \text{ as}$$

n→∞

This implies $\|d(fa,b)\| = 0$

Which implies $fa = b$ **2.2(11)**

Therefore $fa = ga = b$ thus f and g have a coincidence point in X.

Now we shall prove point of coincidence is unique.

Let r be any point of coincidence of f and g Then $fy = gy = r$ **2.2(12)**

Suppose v is any point of coincidence of f and g then we have $fc = gc = v$ **2.2(13)**

Now, $d(r,v) = d(fy,fc) \le k[d(fy,gy) + d(fc,gc)]$

$$\le k[d(fy,fy) + d(fc,fc)] = 0$$

$$d(r,v) = 0$$

$$r = v$$

Thus f and g have unique point of coincidence in X

Since f and g are weakly compatible then from 2.2(12) we have $fr = fgy = gfy = gr$

Therefore $fr = gr = w(say)$

This shows that w is another point of coincidence of f and g

Therefore by the uniqueness of point of coincidence we must have $w = r$

Hence, there exist unique point r ϵ X such that $fr = gr = r$

Thus r is a unique common fixed point of self mappings f and g

Hence f and g have a unique common fixed point

Theorem 2.3: let (X, d) be a cone hexagonal metric space. Suppose the self maps $f, g : X \to X$ satisfy the contractive condition $d(fx, fy) \leq k[d(fx, gy) + d(fy, gx)], \forall x, y \in X$ **2.3(1)**

Where $k \in [0, \frac{1}{2})$ is a constant. If the range of f is contained in the range of g and the range of g is a complete subspace of X then f and g have a unique coincidence point in X more over if f and g are weakly compatible, then f and g have unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary choose $x_1 \in X$ such that $f(x_0) = g(x_1)$ .This is possible since $f(X) \subseteq g(X)$ continuing this process, choose $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$, n= 0, 1, 2…….

Now we define a sequence {y_n} in X such that $y_n = f(x_n) = g(x_{n+1})$ for n=0, 1, 2…… **2.3 (2)**

If $y_m = y_{m+1}$ for some $m \in N$, then $y_m = f(x_{m+1}) = g(x_{m+1})$

That is f and g have a coincidence point $x_{m+1} \in X$

Assume $y_n \neq y_{n+1}, \forall n \in N$

Now $d(y_n, y_{n+1}) = d(fx_n, fx_{n+1})$

$\leq k[d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)]$ From 2.3(1)

$= k[d(y_n, y_n) + d(y_{n+1}, y_{n-1})]$ From 2.3(2)

$$= k[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]$$

Which implies that $d(y_n, y_{n+1}) \leq \frac{k}{1-k} d(y_{n-1}, y_n)$ for n= 0, 1, 2…….

Thus $d(y_n, y_{n+1}) \leq h d(y_{n-1}, y_n)$

$$\leq h^2 d(y_{n-2}, y_{n-1})$$

……

……

$$d(y_n, y_{n+1}) \leq h^n d(y_0, y_1) \qquad \textbf{2.3(3)}$$

For all $n \geq 0$ where $h = \frac{k}{1-k} < 1$

Now $d(y_n, y_{n+2}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})$

$$\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1)$$

$$\leq h^n (1+h) d(y_0, y_1)$$

$$d(y_n, y_{n+2}) \leq \frac{h^n}{1-h} d(y_0, y_1) \qquad \textbf{2.3(4)}$$

Similarly $d(y_n, y_{n+3}) \leq \frac{h^n}{1-h} d(y_0, y_1)$ **2.3(5)**

$$d(y_n, y_{n+4}) \leq \frac{h^n}{1-h} d(y_0, y_1) \qquad \textbf{2.3(6)}$$

Now for the sequence $\{y_n\}$ we consider $d(y_n, y_{n+p})$ in four cases

Case I: p= 4m+1 for m ≥ 1

By hexagonal equality we have

$$d(y_n, y_{n+4m+1}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+4m+1})$$

$$\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5})$$
$$+ d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+7}) + d(y_{n+7}, y_{n+8}) + d(y_{n+8}, y_{n+4m+1})$$

$$= d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) +$$
$$\ldots\ldots\ldots\ldots\ldots + d(y_{n+4m-1}, y_{n+4m}) + d(y_{n+4m}, y_{n+4m+1})$$

$$\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + + h^{n+2} d(y_0, y_1) + \ldots\ldots + h^{n+4m-1} d(y_0, y_1) + h^{n+4m} d(y_0, y_1)$$

$$\leq h^n(1+h+h^2+h^3+..........)d(y_0,y_1)$$

Hence $d(y_n,y_{n+4m+1}) \leq \dfrac{h^n}{1-h} d(y_0,y_1)$, **m ≥ 1** **2.3(7)**

Case II: p= 4m+2 for m ≥ 1

By hexagonal equality we have

$$d(y_n,y_{n+4m+2}) \leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+4m+2})$$

$$\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5}) + d(y_{n+5},y_{n+6})+d(y_{n+6},y_{n+7})+d(y_{n+7},y_{n+8})+d(y_{n+8},y_{n+4m+2})$$

$$= d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5})+ ....+d(y_{n+4m},y_{n+4m+1})+d(y_{n+4m+1},y_{n+4m+2})$$

$$\leq h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)++h^{n+2}d(y_0,y_1)+.......+h^{n+4m}d(y_0,y_1)++h^{n+4m+1}d(y_0,y_1)$$

$$\leq h^n(1+h+h^2+h^3+..........)d(y_0,y_1)$$

Hence $d(y_n,y_{n+4m+2}) \leq \dfrac{h^n}{1-h} d(y_0,y_1)$, **m ≥ 1** **2.3(8)**

Case III: p= 4m+3 for m ≥ 1

By hexagonal equality we have

$$d(y_n,y_{n+4m+3}) \leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+4m+3})$$

$$\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5}) + d(y_{n+5},y_{n+6})+d(y_{n+6},y_{n+7})+d(y_{n+7},y_{n+8})+d(y_{n+8},y_{n+4m+3})$$

$$= d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},y_{n+5})+ ....+d(y_{n+4m+1},y_{n+4m+2})+d(y_{n+4m+2},y_{n+4m+3})$$

$$\leq h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)+h^{n+2}d(y_0,y_1)+.......+h^{n+4m+1}d(y_0,y_1)+h^{n+4m+2}d(y_0,y_1)$$

$$\leq h^n(1+h+h^2+h^3+..........)d(y_0,y_1)$$

Hence $d(y_n,y_{n+4m+3}) \leq \dfrac{h^n}{1-h} d(y_0,y_1)$, **m ≥ 1** **2.3(9)**

Case IV: p= 4m+4 for m ≥ 1

By hexagonal equality we have

$$d(y_n, y_{n+4m+4}) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+4m+4})$$

$$\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + d(y_{n+5}, y_{n+6}) + d(y_{n+6}, y_{n+7}) + d(y_{n+7}, y_{n+8}) + d(y_{n+8}, y_{n+4m+4})$$

$$= d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+5}) + \dots\dots\dots\dots\dots + d(y_{n+4m+2}, y_{n+4m+3}) + d(y_{n+4m+3}, y_{n+4m+4})$$

$$\le h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + + h^{n+2} d(y_0, y_1) + \dots\dots + h^{n+4m+2} d(y_0, y_1) + h^{n+4m+3} d(y_0, y_1)$$

$$\le h^n (1 + h + h^2 + h^3 + \dots\dots\dots\dots) d(y_0, y_1)$$

Hence $d(y_n, y_{n+4m+4}) \le \frac{h^n}{1-h} d(y_0, y_1)$, **m ≥ 1** **2.3(10)**

Thus from above four cases, we have $d(y_n, y_{n+p}) \le \frac{h^n}{1-h} d(y_0, y_1)$

Since P is normal cone we have $\|d(y_n, y_{n+p})\| \le \mu \frac{h^n}{1-h} \|d(y_0, y_1)\| \to 0$ as n→∞ for all p ϵ N

Hence $\{ y_n \}$ is a Cauchy sequence in X.

Since $g(X)$ is a complete subspace of X, there exist b ϵ X such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_{n+1} = b$

Also we can find a ϵ X such that ga=b

Now consider

$$d(fa, b) \le d(fa, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, b)$$

$$= d(fa, fx_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, b)$$

$$\le k[d(fa, gx_{n+1}) + d(fx_{n+1}, ga)] + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, b)$$

$$\le k[d(fa, y_n) + d(y_{n+1}, b)] + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, b)$$

$$\leq k[d(fa,b)+d(b,y_n)]+kd(y_{n+1},b)+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})$$
$$+d(y_{n+3},y_{n+4})+d(y_{n+4},b)$$

$$d(fa,b)\leq\frac{1}{1-k}[kd(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},b)]$$

Since P is normal cone we have

$$\|d(fa,b)\|\leq\frac{\mu}{1-k}\|kd(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+d(y_{n+4},b)\|\to 0$$

As n→∞

This implies $\|d(fa,b)\|=0$

Which implies $fa=b$ **2.3(11)**

Therefore $fa=ga=b$ thus f and g have a coincidence point in X.

Now we shall prove point of coincidence is unique.

Let r be any point of coincidence of f and g Then $fy=gy=r$ **2.3(12)**

Suppose v is any point of coincidence of f and g then we have $fc=gc=v$ **2.3(13)**

Now, $d(r,v)=d(fy,fc)\leq k[d(fy,gc)+d(fc,gy)]$

$$\leq k[d(r,v)+d(v,r)]=0$$

$$d(r,v)=0$$

$$r=v$$

Thus f and g have unique point of coincidence in X

Since f and g are weakly compatible then from 2.3(12) we have $fr=fgy=gfy=gr$

Therefore $fr=gr=w(say)$

This shows that w is another point of coincidence of f and g

Therefore by the uniqueness of point of coincidence we must have $w=r$

Hence, there exist unique point r ϵ X such that $fr=gr=r$

Thus r is a unique common fixed point of self mappings f and g

Hence f and g have a unique common fixed point

To illustrate theorem 2.1, 2.2 and 2.3 we give the following example

Example 2.4: let $X=\{1,2,3,4,5,6\}, E=R^2$ and $P=\{(x,y):x,y\geq 0\}$ is a normal cone in E. Define

d: X × X → E as follows:

$d(1,2)=d(2,1)=(5,10)$

$d(1,3)=d(3,1)=d(1,4)=d(4,1)=d(1,5)=d(5,1)=d(2,3)=d(3,2)=d(2,4)=d(4,2)=d(2,5)=d(5,2)$
$=d(3,4)=d(4,3)=d(3,5)=d(5,3)=d(4,5)=d(5,4)=(1,2)$

$d(1,6)=d(6,1)=d(2,6)=d(6,2)=d(3,6)=d(6,3)=d(4,6)=d(6,4)=d(5,6)=d(6,5)=(4,8)$

Then (X, d) is a complete cone hexagonal metric space

Now we define the self maps $f,g:X\to X$ as follows $f(x)=\begin{cases}3, if\ x\neq 6\\ 1, if\ x=6\end{cases}$ and

$$g(x)=\begin{cases}2, if\ x=1\\ 1, if\ x=2\\ 3, if\ x=3\\ 5, if\ x=4\\ 4, if\ x=5\\ 6, if\ x=6\end{cases}$$

It is clear that $f(x)\subseteq g(x)$, f and g are weakly compatible, clearly f and g satisfies the contractive conditions 2.1(1) 2.2(1) and 2.3(1) of theorems 2.1, 2.2 and2.3. Hence f and g satisfies all the conditions of theorems of 2.1, 2.2 and 2.3. And 3 is unique common fixed point f and g

References:

[1]. B.E.Rhoades, “A comparison of various definitions of contractive mappings”, *Trans.Amer.Math.Soc.*26 (1977) 257-290.

[2]. Paula Collaco and Jaime Carvalho E Silva, “A complete comparison of 25 contraction conditions”, *Nonlinear Analysis, Theory, Methods and Applications,* Vol.30, No.1, pp. 471-476.

[3]. L.G.Huang, X.Zhang, “Cone metric spaces and fixed point theorems of contractive mappings”, *J.Math.Anal.Appl.*332 (2) (2007) 1468-1476.

[4]. M.Abbas and G. Jungck, "Common fixed point results for non-commutating mappings without

Continuity in cone metric spaces", *J.Math.Anal.Appl.* 341(2008)416-420.

[5]. M.Abbas and B.E.Rhoades, "Fixed and Periodic point results in Cone metric spaces", *Appl.Math.Lett.*22 (2009), 511-515.

[6]. S.Rezapour and Halmbarani, "Some notes on the paper "cone metric spaces and fixed point theorems

Of contractive mappings", *J.Math.Anal.Appl.345* (2008)719-729.

[7]. S.Rezapour, R.H.Hagi, "Fixed points of multi functions on cone metric spaces", *Number.Funct.Anal*

. 30(7-9)(2009)825-832.

[8]. Stojan Radenovic, "Common fixed points under contractive conditions in cone metric spaces",

Computers and Mathematics with Applications. 58(2009)1273-1278.

[9]. Vetro, "Common fixed points in cone metric spaces", *Rend.Ciric. Mat.Palermo.* 56(2007)464-468.

[10]. T.Abdeljawad, E.Karapinar and K.Tas, "Common fixed point theorems in cone Banach spaces",

Hacet. J.Math.Stat. 40(2011), 211-217.

[11]. Jankovic, S.Dadelburg, Z.Radenovic, "On cone metric spaces, A survey", *Nonlinear Analysis,*

74(2011), 2591-2601.

[12]. P.Raja, S.M.Vaezpour, "Some extensions of Banach's contraction principle in complete cone metric

Spaces", *Fixed Point Theory and Applications,* Article ID768294 (2008), 11pages

[13]. Manoj Garg and Shikha Agarwal; Banach contraction principal on cone pentagonal metric spaces,

J. Adv. Stud. Topo, 3(1), 12 - 18 (2012

[14]. Manoj Garg "Banach Contraction Principle on Cone Hexagonal Metric Space**,** Ultra Scientist Vol

. 26(1) A, 97-103 (2014).