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ON THE DEGENERATE LAPLACE TRANSFORM - II

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Abstract

We continue our previous study [19] in advancement of the very recent study of Kim and Kim [18]. In this second paper of this series we propose to establish the results for the degenerate Laplace transform of an integral, the expression for the division theorem in case of the degenerate Laplace transform, the degenerate Laplace transform of a periodic function, the initial and final value theorems and we also give some illustrations for the first translation theorem of degenerate Laplace transforms which was proved by us in our previous study [19] of this topic.

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1. Introduction

One of the most widely used tools of mathematical analysis in the applied sciences, physics and engineering is the Laplace transform, named after its discover Pierre Simon Laplace. This technique mainly involves the transforming (converting) of a function of a real variable (which, in the context of electrical and electronics engineering problems, may usually be the time variable 't') into the function of a complex variable (which, in the engineering problems may be the frequency variable 's'). Because most of the functions for which we take the Laplace transform are often considered to be defined only in the region $t \geq 0$ for the real variable t, therefore, whenever the Laplace transform of such a function exits, then it is a holomorphic function of the complex variable (parameter of the Laplace transform) s. The widespread applications of the Laplace transform in various

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branches of study make it one of the most widely studied fields of research in mathematics. To mention just a few, this integral transform has been used extensively in finding the solutions to a wide variety of problems of applied mathematics, physics, electrical engineering, control theory, nuclear physics, statistics, mechanical engineering, astronomical problems, etc. Given such importance of the classical Laplace transform concept, this concept was generalized in a number of ways by different researchers at different times. A recent q-extension of the Laplace transform is due to Chung, Kim and Kwon [10]. The Laplace transform method approach is a well settled approach for defining the special functions of matrix arguments, especially the multiple hypergeometric functions of matrix arguments, when the argument matrices of these functions are real symmetric positive definite matrices or complex Hermitian positive definite matrices. The most authoritative work till date in this direction is the work of Mathai [9, (Chapters 5 and 6)]. A number of Laplace type integrals for Appell's and Humbert's functions of matrix arguments with real symmetric positive definite matrices as arguments can be found in the works of this author [4-8]. These integrals generalize the corresponding results for these functions of scalar arguments which already existed in the literature prior to these studies of this author. The most recent generalization of the classical Laplace transform concept is called the 'degenerate Laplace transform' which owes its origin in the very recent work of Kim and Kim [18]. We mention here that prior to giving this concept of degenerate Laplace transform T. Kim and his coworkers D.V. Dolgy, J.J. Seo, L.C. Jang, H.-I. Kwon and D.S. Kim have made extensive studies on degenerate poynomials and numbers [11-18]. The outstanding works of Carlitz in this context also deserve to be mentioned [1,2]. In this paper we propose to establish some more properties of the degenerate Laplace transform, in succession to our previous study [19]. We utilize the definitions and results of the most recent study of Kim and Kim [18] to establish the results for the degenerate Laplace transform of an integral, the expression for the division theorem in the case of the degenerate Laplace transform, the degenerate Laplace transform of a periodic function, the initial and final value theorems for the degenerate Laplace transforms and also give some illustrations of the first translation theorem for the degenerate Laplace transforms proved by us in our previous study [19, Theorem 2.1]. The paper consists of two sections. The brief outline of our paper is that we give the necessary preliminary definitions and concepts in the first section of the paper, while these concepts and definitions will be utilized by us in the second section of the paper to derive the aforesaid results.

Now we proceed to give the preliminary definitions and results and we mention that we follow the notations used by Kim and Kim [18] for denoting the Laplace transform and the degenerate Laplace transform of a function throughout this paper. The Laplace transform of a function $f(t)$ of the variable t defined for $t>0$, denoted by $\mathcal{L}\{f(t)\}$, is defined by the integral (see, for instance, [3, (1), p.1])

$$\mathcal{L}\{f(t)\}=F(s)=\int_0^\infty e^{-st}f(t)\,dt \tag{1.1}$$

provided the integral in (1.1) converges for some value of the complex parameter s. For sufficient conditions for the existence of the Laplace transform of a function $f(t)$, the reader is referred to the Theorem 1-1, p.2 and the Problem 145, p.38 of [3].

Definition 1.1: The Degenerate Exponential Function - (Kim and Kim [18], (1.3), p. 241) – The degenerate exponential function, represented by e_λ^t, is a function of two variables λ and t, where, $\lambda \in (0,\infty)$, $t \in \mathbb{R}$ and is defined by

$$e_\lambda^t = (1 + \lambda t)^{\frac{1}{\lambda}} \tag{1.2}$$

It may be noted here that this definition generalizes the classical exponential function e^t defined by the well known series relation

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{1.3}$$

because we can easily deduce from (1.2) that (see Kim and Kim [18], p.241)

$$\lim_{\lambda \to 0+} e_\lambda^t = \lim_{\lambda \to 0+} (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t \tag{1.4}$$

The well known Euler's exponential formula is given by (see, for instance, (1.7) p.241 Kim and Kim [18])

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.5}$$

where $i = \sqrt{-1}$. From this follow immediately the definitions of the elementary trigonometric functions sine and cosine in terms of the exponential function as (see, for instance, (1.8) p.241 Kim and Kim[18])

$$\cos a\theta = \frac{e^{ia\theta} + e^{-ia\theta}}{2}, \quad \sin a\theta = \frac{e^{ia\theta} - e^{-ia\theta}}{2i} \tag{1.6}$$

Definition 1.2: The Degenerate Euler Formula: The degenerate Euler formula is defined by the relation (see (1.9), p.242 Kim and Kim[18])

$$e_\lambda^{it} = (1 + \lambda t)^{\frac{i}{\lambda}} = \cos_\lambda(t) + i\sin_\lambda(t) \tag{1.7}$$

From (1.7) we can be readily infer that (see (1.10) p.242 Kim and Kim [18])

$$\lim_{\lambda \to 0+} e_\lambda^{it} = \lim_{\lambda \to 0+} (1 + \lambda t)^{\frac{i}{\lambda}} = e^{it} = \cos t + i\sin t \tag{1.8}$$

Further it is rather easy to note from (1.7) and (1.8) that (see (1.11) p.242 Kim and Kim [18])

$$\lim_{\lambda \to 0+} \cos_\lambda(t) = \cos t, \quad \lim_{\lambda \to 0+} \sin_\lambda(t) = \sin t \tag{1.9}$$

Definition 1.3: The Degenerate Cosine and Degenerate Sine Functions: The following respective definitions of the degenerate cosine and degenerate sine functions follow from (1.7) (see (1.12), p.242 Kim and Kim[18])

$$\cos_\lambda(t) = \frac{e_\lambda^{it} + e_\lambda^{-it}}{2}, \quad \sin_\lambda(t) = \frac{e_\lambda^{it} - e_\lambda^{-it}}{2i} \tag{1.10}$$

Definition 1.4: The Degenerate Laplace Transform of a Function: (Kim and Kim [18], (3.1), p.244) Let $f(t)$ be a function defined for $t \geq 0$ and let $\lambda \in (0,\infty)$, then the degenerate Laplace transform of the function $f(t)$, represented by $F_\lambda(s)$, is defined by the integral

$$\mathcal{L}_\lambda\{f(t)\} = F_\lambda(s) = \int_0^\infty (1 + \lambda t)^{\frac{-s}{\lambda}} f(t)\,dt \tag{1.11}$$

2. Various Results for the Degenerate Laplace Transform of a Function

In this section we proceed to derive a number of results, as stated above and in the abstract of the paper for the degenerate Laplace transform of a function. The results will be stated as theorems and proved by the use of the concepts stated above.

Theorem 2.1: Degenerate Laplace Transform of an Integral:

Let $\mathcal{L}_\lambda\{f(t)\} = F_\lambda(s)$ then

$$\mathcal{L}_\lambda\{f(t)\} = s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}\int_0^t f(u)\,du\right\} \qquad (2.1)$$

Proof: Let $g(t) = \int_0^t f(u)\,du$ then $g'(t) = \frac{d}{dt}\left\{\int_0^t f(u)\,du\right\} = f(t)$ and $g(0)=0$. Kim and Kim [18] (see (3.20), p 246) have shown that

$$\mathcal{L}_\lambda\{f'(t)\} = \mathcal{L}_\lambda\left\{\frac{d}{dt}f(t)\right\} = -f(0) + s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\}$$

Applying this result to the function $g(t)$ gives

$$\mathcal{L}_\lambda\{g'(t)\} = -g(0) + s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}g(t)\right\}$$

(2.1) now directly follows from this relation on noting that $g'(t)=f(t)$ and $g(0)=0$. It may be pointed out here that this theorem generalizes the Theorem 1-11 p.4 of [3], to which it reduces in the limiting case $\lambda \to 0+$.

Theorem 2.2: Division by t : If $\mathcal{L}_\lambda\{f(t)\} = F_\lambda(s)$ then

$$\int_s^\infty F_\lambda(s)\,ds = \lambda\mathcal{L}_\lambda\left\{\frac{f(t)}{\log(1+\lambda t)}\right\} \qquad (2.2)$$

Proof: From (1.11) follows

$$F_\lambda(s) = \mathcal{L}_\lambda\{f(t)\} = \int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)\,dt$$

Integrating both sides of this equation with respect to the variable s from $s=s$ to $s=\infty$ we have

$$\int_s^\infty F_\lambda(s)\,ds = \int_s^\infty ds\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)\,dt$$

Since s and t are independent variables in this case, therefore, we can interchange the order of integration in the above double integral to obtain

$$\int_s^\infty F_\lambda(s)ds = \int_0^\infty dt\int_s^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)ds = \int_0^\infty f(t)dt\left[\int_s^\infty (1+\lambda t)^{\frac{-s}{\lambda}}ds\right]$$

$$= \int_0^\infty f(t)dt\left[\frac{(1+\lambda t)^{\frac{-s}{\lambda}}}{\log(1+\lambda t)^{\frac{-1}{\lambda}}}\right]_{s=s}^{\infty} = \int_0^\infty f(t)dt\left[0 - \frac{(1+\lambda t)^{\frac{-s}{\lambda}}}{\frac{-1}{\lambda}\log(1+\lambda t)}\right]$$

$$= \lambda\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}\frac{f(t)}{\log(1+\lambda t)}dt = \lambda\mathcal{L}_\lambda\left\{\frac{f(t)}{\log(1+\lambda t)}\right\}.$$

This theorem generalizes the Theorem 1-13 p.5 of [3] and to which it reduces in the limiting case $\lambda \to 0+$.

Theorem 2.3: Degenerate Laplace Transform of Periodic Functions: Suppose $f(t)$ be a periodic function with period ω, i.e. $f(t+n\omega) = f(t)$ for $n=0,1,2,\ldots,$ then,

$$\mathcal{L}_\lambda\left\{f(t)\right\}=\sum_{n=0}^{\infty}(1+\lambda n\omega)^{\frac{-s}{\lambda}}\left[\int_0^{\omega}(1+\alpha u)^{\frac{-s}{\alpha}}(1+\alpha u)^{sn\omega}f(u)du\right] \quad (2.3)$$

where, $\alpha=\dfrac{\lambda}{1+\lambda n\omega}$.

Proof: By the definition of the degenerate Laplace transform, i.e. (1.11) we have

$$\mathcal{L}_\lambda\left\{f(t)\right\}=\int_0^{\infty}(1+\lambda t)^{\frac{-s}{\lambda}}f(t)dt=\sum_{n=0}^{\infty}\int_{n\omega}^{(n+1)\omega}(1+\lambda t)^{\frac{-s}{\lambda}}f(t)dt$$

$$=\sum_{n=0}^{\infty}\int_0^{\omega}\left(1+\lambda(u+n\omega)\right)^{\frac{-s}{\lambda}}f(u+n\omega)du, \qquad \text{where } t=u+n\omega$$

$$=\sum_{n=0}^{\infty}\int_0^{\omega}(1+\lambda n\omega)^{\frac{-s}{\lambda}}\left(1+\frac{\lambda u}{1+\lambda n\omega}\right)^{\frac{-s}{\lambda}}f(u)du$$

$$=\sum_{n=0}^{\infty}(1+\lambda n\omega)^{\frac{-s}{\lambda}}\int_0^{\omega}(1+\alpha u)^{\frac{-s}{\alpha}(1-\alpha n\omega)}f(u)du \qquad \text{where } \alpha=\frac{\lambda}{1+\lambda n\omega}$$

$$=\sum_{n=0}^{\infty}(1+\lambda n\omega)^{\frac{-s}{\lambda}}\left[\int_0^{\omega}(1+\alpha u)^{\frac{-s}{\alpha}}(1+\alpha u)^{sn\omega}f(u)du\right]$$

which proves (2.3). This theorem generalizes the Theorem 1-14, p.5 of [3]. When $\lambda\to 0+$ in this theorem, it reduces to the Theorem 1-14, p.5 of [3].

Theorem 2.4: Initial Value Theorem:

$$\lim_{t\to 0}f(t)=\lim_{s\to\infty}s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\} \quad (2.4)$$

Proof: We start with the following result of Kim and Kim [18] (see (3.20), p. 246)

$$\mathcal{L}_\lambda\left\{f'(t)\right\}=\mathcal{L}_\lambda\left\{\frac{d}{dt}f(t)\right\}=-f(0)+s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\}$$

Or,

$$\int_0^{\infty}(1+\lambda t)^{\frac{-s}{\lambda}}f'(t)dt=-f(0)+s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\} \quad (2.5)$$

Taking limits of this expression as $s\to\infty$, we get

$$\lim_{s\to\infty}\int_0^{\infty}(1+\lambda t)^{\frac{-s}{\lambda}}f'(t)dt=-f(0)+\lim_{s\to\infty}s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\}$$

which yields

$$0=-f(0)+\lim_{s\to\infty}s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\}$$

from where (2.4) directly follows by noting that $f(0)=\lim_{t\to 0}f(t)$. This theorem generalizes the Theorem 1-16 p.5 of [3], to which it reduces when $\lambda\to 0+$.

Theorem 2.5: Final Value Theorem:

$$\lim_{t\to\infty}f(t)=\lim_{s\to 0}s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\} \quad (2.6)$$

Proof: We take the limit of (2.5) as $s\to 0$ to obtain

$$\int_0^{\infty}f'(t)dt=-f(0)+\lim_{s\to 0}s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}f(t)\right\}$$

Or,

$$\left[f(t)\right]_{t=0}^{\infty} = -f(0) + \lim_{s\to 0} s\mathcal{L}_{\lambda}\left\{(1+\lambda t)^{-1} f(t)\right\}$$

which yields precisely (2.7) on minor simplification. This theorem generalizes the Theorem 1-17 p.6 of [3], which is the limiting case of this theorem as $\lambda \to 0+$.

2.6: Illustrations for the First Translation Theorem of Degenerate Laplace Transform – We have shown in [19, Theorem 2.1] if $\mathcal{L}_{\lambda}\left\{f(t)\right\} = F_{\lambda}(s)$, then

$$\mathcal{L}_{\lambda}\left\{e_{\lambda}^{at} f(t)\right\} = F_{\lambda}(s-a) \tag{2.7}$$

This result is known as the first translation theorem of the degenerate Laplace transform. We give the following examples to illustrate this theorem. Kim and Kim [18] (see (3.10) and (3.11), p. 245 of [18]) have defined the *degenerate hyperbolic sine* and the *degenerate hyperbolic cosine* functions by the relations

$$\cosh_{\lambda}(at) = \frac{1}{2}\left\{(1+\lambda t)^{\frac{a}{\lambda}} + (1+\lambda t)^{\frac{-a}{\lambda}}\right\} \tag{2.8}$$

$$\sinh_{\lambda}(at) = \frac{1}{2}\left\{(1+\lambda t)^{\frac{a}{\lambda}} - (1+\lambda t)^{\frac{-a}{\lambda}}\right\} \tag{2.9}$$

They have also shown that (see [18, (3.14) and (3.15) p.245])

$$\mathcal{L}_{\lambda}\left\{\cosh_{\lambda}(at)\right\} = \frac{s-\lambda}{(s-\lambda)^2 - a^2} \tag{2.10}$$

$$\mathcal{L}_{\lambda}\left\{\sinh_{\lambda}(at)\right\} = \frac{a}{(s-\lambda)^2 - a^2} \tag{2.11}$$

From the relations (2.7), (2.10), (2.11) it immediately follows that

$$\mathcal{L}_{\lambda}\left\{e_{\lambda}^{at}\cosh_{\lambda}(bt)\right\} = \frac{s-a-\lambda}{(s-a-\lambda)^2 - b^2} \tag{2.12}$$

and

$$\mathcal{L}_{\lambda}\left\{e_{\lambda}^{at}\sinh_{\lambda}(bt)\right\} = \frac{b}{(s-a-\lambda)^2 - b^2} \tag{2.13}$$

For their degenerate Laplace transform Kim and Kim [18, p. 244] have stated the linearity property as

$$\mathcal{L}_{\lambda}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha\mathcal{L}_{\lambda}\left\{f(t)\right\} + \beta\mathcal{L}_{\lambda}\left\{g(t)\right\}, \qquad \text{where } \alpha,\beta \in \mathbb{R} \tag{2.14}$$

Using (2.12) – (2.14) it can be readily inferred that

$$\begin{aligned}
&\mathcal{L}_{\lambda}\left\{e_{\lambda}^{-t}\left(3\sinh_{\lambda}(2t) - 5\cosh_{\lambda}(2t)\right)\right\} \\
&= 3\mathcal{L}_{\lambda}\left\{e_{\lambda}^{-t}\sinh_{\lambda}(2t)\right\} - 5\mathcal{L}_{\lambda}\left\{e_{\lambda}^{-t}\cosh_{\lambda}(2t)\right\} \\
&= 3\left\{\frac{2}{(s+1-\lambda)^2 - 4}\right\} - 5\left\{\frac{(s+1-\lambda)}{(s+1-\lambda)^2 - 4}\right\} \\
&= \frac{(1-5s+5\lambda)}{(s+1-\lambda)^2 - 4}
\end{aligned}$$

As another illustration, we can write with the help of (2.7) and (2.11) that

$$\mathcal{L}_\lambda\left\{e_\lambda^{iat}\sinh_\lambda(at)\right\}=\frac{a}{(s-ia-\lambda)^2-a^2}$$

$$=\frac{a\left\{(s-\lambda)^2-2a^2+2ia(s-\lambda)\right\}}{(s-\lambda)^4+4a^4}$$

With the help of (1.7) this equation may be rewritten as

$$\mathcal{L}_\lambda\left\{\left(\cos_\lambda(at)+i\sin_\lambda(at)\right)\sinh_\lambda(at)\right\}=\frac{a\left\{(s-\lambda)^2-2a^2+2ia(s-\lambda)\right\}}{(s-\lambda)^4+4a^4}$$

which, on equating the real and imaginary parts at once yields

$$\mathcal{L}_\lambda\left\{\cos_\lambda(at)\sinh_\lambda(at)\right\}=\frac{a\left\{(s-\lambda)^2-2a^2\right\}}{(s-\lambda)^4+4a^4}$$

and

$$\mathcal{L}_\lambda\left\{\sin_\lambda(at)\sinh_\lambda(at)\right\}=\frac{2a^2(s-\lambda)}{(s-\lambda)^4+4a^4}$$

We conclude the paper by mentioning that the author shall communicate more results on the degenerate Laplace transforms in the very next communication within a few days from now.

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