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ON THE DEGENERATE LAPLACE TRANSFORM - IV

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Abstract

The present paper is in progression of our earlier studies [19-21] of the degenerate Laplace transform and in furtherance of the very recent study of Kim and Kim [18]. Here we aim at defining the degenerate Bessel function, the degenerate exponential integral function and the degenerate error function and two new variants - one each for the degenerate sine integral and the degenerate cosine integral and also obtain the degenerate Laplace transforms of all these functions.

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1. Introduction

Many important problems of applied mathematics, physics and engineering depend upon an extensive knowledge of the mathematical tools. The Laplace transform is one such inevitable tool and finds prominent place in all the books of mathematics which are specifically designed for engineers, physicists and applied mathematicians, see for instance, [23, Chapter 6]. The special functions of mathematics are also indispensable for the study of solutions of many problems arising in engineering and physics. Not only these special functions but, we also require the Laplace transforms of these functions for solving the various problems which occur frequently in these domains. Due to this fact many standard and authoritative works in mathematics, e.g. [24-28] to mention a few, are devoted solely to collecting and tabulating the Laplace transforms as well as the

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other integral transforms of the various special functions which are widely scattered in the literature. These handbooks are used extensively by all those researchers working in other fields like engineering, physics, applied mathematics, statistics and social sciences, etc., who are often confronted with the various special functions and their properties in their respective areas of works. With these diverse applications in the mind, we propose to define the degenerate Bessel function, the degenerate exponential integral function and the degenerate error function and give two new variants of the degenerate sine integral and the degenerate cosine integral and further to obtain their degenerate Laplace transforms in this study. The concept of degenerate Laplace transform was very recently introduced by Kim and Kim [18] and we have, in our three earlier studies [19-21] on this topic made a humble attempt to extend their novel work. The most recent works which can be cited in this and related fields are those of T. Kim and his coworkers D.V. Dolgy, J.J. Seo, L.C. Jang, H.-I. Kwon and D.S. Kim all of whom have studied the degenerate polynomials and numbers [11-18]. The Laplace transforms of some multiple hypergeometric functions of matrix arguments with real symmetric positive definite matrices and Hermitian positive definite matrices as arguments can be found in the works [4-9]. We appeal to the results of Kim and Kim [18] to establish our results in this paper. This paper consists of two sections. The first section contains the necessary background material for the purpose of development of the results in the second part of the paper.

The Laplace transform of a function $f(t)$ of the variable t defined for $t>0$, denoted by $\mathcal{L}\{f(t)\}$, is defined by the integral (see, for instance, [3, (1), p.1])

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{1.1}$$

provided the integral in (1.1) converges for some value of the complex parameter s. For sufficient conditions for the existence of the Laplace transform of a function $f(t)$, the reader is referred to the Theorem 1-1, p.2 and the Problem 145, p.38 of [3].

Definition 1.1: The Degenerate Exponential Function - (Kim and Kim [18], (1.3), p. 241) – The degenerate exponential function, represented by e_λ^t, is a function of two variables λ and t, where, $\lambda \in (0,\infty)$, $t \in \mathbb{R}$ and is defined by

$$e_\lambda^t = (1+\lambda t)^{\frac{1}{\lambda}} \tag{1.2}$$

It may be noted here that this definition generalizes the classical exponential function e^t defined by the well known series relation

$$e^t = \sum_{n=0}^\infty \frac{t^n}{n!} \tag{1.3}$$

because we can easily deduce from (1.2) that (see Kim and Kim [18], p.241)

$$\lim_{\lambda\to 0+} e_\lambda^t = \lim_{\lambda\to 0+} (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^\infty \frac{t^n}{n!} = e^t \tag{1.4}$$

The well known Euler's exponential formula is given by (see, for instance, (1.7) p.241 Kim and Kim [18])

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.5}$$

where $i=\sqrt{-1}$. From this follow immediately the definitions of the elementary trigonometric functions sine and cosine in terms of the exponential function as (see, for instance, (1.8) p.241 Kim and Kim[18])

$$\cos a\theta = \frac{e^{ia\theta} + e^{-ia\theta}}{2}, \quad \sin a\theta = \frac{e^{ia\theta} - e^{-ia\theta}}{2i} \tag{1.6}$$

Definition 1.2: The Degenerate Euler Formula: The degenerate Euler formula is defined by the relation (see (1.9), p.242 Kim and Kim[18])

$$e_\lambda^{it} = (1+\lambda t)^{\frac{i}{\lambda}} = \cos_\lambda(t) + i\sin_\lambda(t) \tag{1.7}$$

From (1.7) we can be readily infer that (see (1.10) p.242 Kim and Kim [18])

$$\lim_{\lambda \to 0+} e_\lambda^{it} = \lim_{\lambda \to 0+} (1+\lambda t)^{\frac{i}{\lambda}} = e^{it} = \cos t + i\sin t \tag{1.8}$$

Further it is rather easy to note from (1.7) and (1.8) that (see (1.11) p.242 Kim and Kim [18])

$$\lim_{\lambda \to 0+} \cos_\lambda(t) = \cos t, \quad \lim_{\lambda \to 0+} \sin_\lambda(t) = \sin t \tag{1.9}$$

Definition 1.3: The Degenerate Cosine and Degenerate Sine Functions: The following respective definitions of the degenerate cosine and degenerate sine functions follow from (1.7) (see (1.12), p.242 Kim and Kim[18])

$$\cos_\lambda(t) = \frac{e_\lambda^{it} + e_\lambda^{-it}}{2}, \quad \sin_\lambda(t) = \frac{e_\lambda^{it} - e_\lambda^{-it}}{2i} \tag{1.10}$$

Definition 1.4: The Degenerate Gamma Function: (Kim and Kim [18], (2.1), p.242) According to Kim and Kim [18] the degenerate Gamma function is defined for each $\lambda \in (0,\infty)$ and for the complex variable s, where, $\left(0 < \mathrm{Re}(s) < \frac{1}{\lambda}\right)$, by the relation

$$\Gamma_\lambda(s) = \int_0^\infty (1+\lambda t)^{-\frac{1}{\lambda}} t^{s-1} dt = \int_0^\infty e_\lambda^{-t} t^{s-1} dt \tag{1.11}$$

They have also shown that (see (2.2), p.242 [18])

$$\Gamma_\lambda(s) = \lambda^{-s} B\left(s, \frac{1}{\lambda} - s\right) \tag{1.12}$$

where, $B(x,y)$ is the well known Beta function defined by (see, for instance, [22], (2) and (5) , §1.5, p.9)

$$B(x,y) = \int_0^\infty \frac{v^{x-1}}{(1+v)^{x+y}} dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \mathrm{Re}(x), \mathrm{Re}(y) > 0 \tag{1.13}$$

Kim and Kim [18] have also shown that (see Theorem 2.1, p. 243 [18]) for $\lambda \in (0,1)$ and for $0 < \mathrm{Re}(s) < \frac{1-\lambda}{\lambda}$

$$\Gamma_\lambda(s+1) = \frac{s}{(1-\lambda)^{s+1}} \Gamma_{\frac{\lambda}{1-\lambda}}(s) \tag{1.14}$$

It is easy to infer from (1.11) that when $\lambda \to 0+$, then the degenerate gamma function reduces to the classical gamma function given by (for example, see, (1), §1.1, p.1 [22])

$$\lim_{\lambda \to 0+} \Gamma_\lambda(s) = \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \qquad \mathrm{Re}(s) > 0 \tag{1.15}$$

Definition 1.5: The Degenerate Laplace Transform of a Function: (Kim and Kim [18], (3.1), p.244) Let $f(t)$ be a function defined for $t \geq 0$ and let $\lambda \in (0,\infty)$, then the degenerate Laplace transform of the function $f(t)$, represented by $F_\lambda(s)$, is defined by the integral

$$\mathcal{L}_\lambda\{f(t)\} = F_\lambda(s) = \int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}} f(t)\,dt \tag{1.16}$$

2. The Degenerate Laplace Transform of Some Special Functions

We carry on now the study of degenerate Laplace transforms of the degenerate Bessel function, the degenerate exponential function, the degenerate error function and the two new variants of the degenerate sine integral and the degenerate cosine integral in this section.

Definition 2.1: The Degenerate Bessel Function: We define the *degenerate Bessel function* by the following relation:

$$J_{n(\lambda)}(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma_\lambda(r+1)\Gamma_\lambda(n+r+1)}\left(\frac{t}{2}\right)^{n+2r} \tag{2.1}$$

We give below the degenerate Laplace transform of the above function.

Theorem 2.2: Degenerate Laplace Transform of the Degenerate Bessel Function:
For $0<\lambda<1$ we have

$$\mathcal{L}_\lambda\{J_{n(\lambda)}(t)\} = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)(1-\lambda)^{n+r+1}}{\Gamma_\lambda(r+1)(n+r)\left(1-\frac{\lambda}{s}\right)^{n+2r+1}}\left(\frac{1}{2}\right)^{n+2r} \frac{\Gamma_{\frac{\lambda}{s-\lambda}}(n+2r)}{\Gamma_{\frac{\lambda}{1-\lambda}}(n+r)} s^{-n-2r-1} \tag{2.2}$$

Proof: We take the degenerate Laplace transform of both sides of (2.1) (keeping in mind the linearity property of the degenerate Laplace transform) to get

$$\mathcal{L}_\lambda\{J_{n(\lambda)}(t)\} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma_\lambda(r+1)\Gamma_\lambda(n+r+1)}\left(\frac{1}{2}\right)^{n+2r} \mathcal{L}_\lambda\{t^{n+2r}\} \tag{2.3}$$

Kim and Kim [18] have shown that (see Theorem 3.2, p.246 in [18]) $n \in \mathbb{N}$ and $s>(n+1)\lambda$

$$\mathcal{L}_\lambda\{t^n\} = s^{-n-1}\Gamma_{\frac{\lambda}{s}}(n+1) \tag{2.4}$$

Thus (2.3), with the aid of (2.4) yields,

$$\mathcal{L}_\lambda\{J_{n(\lambda)}(t)\} = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma_\lambda(r+1)\Gamma_\lambda(n+r+1)}\left(\frac{1}{2}\right)^{n+2r} s^{-n-2r-1}\Gamma_{\frac{\lambda}{s}}(n+2r+1) \tag{2.5}$$

where, $s>(n+2r+1)\lambda$.

With the use of (1.14) we can write

$$\Gamma_{\frac{\lambda}{s}}(n+2r+1) = \frac{n+2r}{\left(1-\frac{\lambda}{s}\right)^{n+2r+1}}\Gamma_{\frac{\lambda}{s-\lambda}}(n+2r)$$

for $\lambda \in (0,1)$ and $0<n+2r<\frac{s-\lambda}{\lambda}$, as well as

$$\Gamma_\lambda(n+r+1)=\frac{n+r}{(1-\lambda)^{n+r+1}}\Gamma_{\frac{\lambda}{1-\lambda}}(n+r)$$

for $\lambda\in(0,1)$ and $0<n+r<\frac{1-\lambda}{\lambda}$.

Now $n+r<\frac{1-\lambda}{\lambda}\Rightarrow n+2r<\frac{1-\lambda}{\lambda}+r$, if we choose s such that $n+2r<\frac{1-\lambda}{\lambda}+r<\frac{s-\lambda}{\lambda}$, i.e. $r<\frac{s-1}{\lambda}$ then it follows from the last two equations that

$$\frac{\Gamma_{\frac{\lambda}{s}}(n+2r+1)}{\Gamma_\lambda(n+r+1)}=\frac{(n+2r)(1-\lambda)^{n+r+1}\Gamma_{\frac{\lambda}{s-\lambda}}(n+2r)}{(n+r)\left(1-\frac{\lambda}{s}\right)^{n+2r+1}\Gamma_{\frac{\lambda}{1-\lambda}}(n+r)} \tag{2.6}$$

for $r<\frac{s-1}{\lambda}$. Substituting from (2.6) in (2.5) now leads us to the result of (2.2).

Definition 2.3: The Degenerate Exponential Function: The *degenerate exponential function* is defined by us in the following manner:

$$\mathrm{Ei}_\lambda(t)=\int_t^\infty\frac{\lambda e_\lambda^{-u}}{\log(1+\lambda u)}du \tag{2.7}$$

It may be noted here that when $\lambda\to0+$ in (2.7) then it reduces to the *exponential function* (see (7), Appendix C, p.255 [3]) given by

$$\mathrm{Ei}(t)=\int_t^\infty\frac{e^{-u}}{u}du \tag{2.8}$$

Below we prove a theorem for the degenerate Laplace transform of the *degenerate exponential function*.

Theorem 2.4: The Degenerate Laplace Transform of the Degenerate Exponential Function: For $s>-1+\lambda$,

$$\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1}\mathrm{Ei}_\lambda(t)\right\}=\frac{1}{s}\log\left(\frac{s+1-\lambda}{1-\lambda}\right) \tag{2.9}$$

Proof: Let $f(t)=\mathrm{Ei}_\lambda(t)=\int_t^\infty\frac{\lambda e_\lambda^{-u}}{\log(1+\lambda u)}du$, i.e.,

$$f(t)=-\int_\infty^t\frac{\lambda e_\lambda^{-u}}{\log(1+\lambda u)}du\Rightarrow\lim_{t\to\infty}f(t)=0\qquad\text{and}\qquad f'(t)=\frac{-\lambda e_\lambda^{-t}}{\log(1+\lambda t)}$$

Or,

$$\log(1+\lambda t)f'(t)=-\lambda e_\lambda^{-t} \tag{2.10}$$

Taking the degenerate Laplace transform of both sides of (2.10) we have

$$\mathcal{L}_\lambda\left\{\log(1+\lambda t)f'(t)\right\}=\mathcal{L}_\lambda\left\{-\lambda e_\lambda^{-t}\right\} \tag{2.11}$$

Kim and Kim [18] have shown that (see Theorem 3.4, p.247 in [18]) for $n\in\mathbb{N}$

$$\mathcal{L}_\lambda\left\{\left[\log(1+\lambda t)\right]^n f(t)\right\}=(-1)^n\lambda^n\frac{d^n}{ds^n}\left\{F_\lambda(s)\right\} \tag{2.12}$$

where, $\mathcal{L}_\lambda\left\{f(t)\right\}=F_\lambda(s)$. Kim and Kim [18] have also deduced that (see (3.20), p.246 [18])

$$\mathcal{L}_\lambda\{f'(t)\} = \mathcal{L}_\lambda\left\{\frac{d}{dt}f(t)\right\} = -f(0) + s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\} \tag{2.13}$$

Using (2.12) in (2.11) for $n=1$ leads us to

$$(-1)\lambda\frac{d}{ds}\left[\mathcal{L}_\lambda\{f'(t)\}\right] = -\lambda\mathcal{L}_\lambda\left\{e_\lambda^{-t}\right\} \tag{2.14}$$

Applying (2.13) and (1.2) to (2.14) yields

$$\frac{d}{ds}\left[-f(0) + s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\}\right] = \mathcal{L}_\lambda\left\{(1+\lambda t)^{\frac{-1}{\lambda}}\right\} \tag{2.15}$$

Kim and Kim [18] have shown that (see (3.3), p.244 in [18])

$$\mathcal{L}_\lambda\left\{(1+\lambda t)^{\frac{-a}{\lambda}}\right\} = \frac{1}{s+a-\lambda}, \qquad \text{if} \qquad s > -a+\lambda \tag{2.16}$$

Noting that $f(0) = \text{constant}$ and using (2.16) in (2.15) produces for $s > -1+\lambda$,

$$\frac{d}{ds}\left[s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\}\right] = \frac{1}{s+1-\lambda} \tag{2.17}$$

Integrating (2.17) with respect to s gives

$$s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\} = \log(s+1-\lambda) + C \tag{2.18}$$

where C is the constant of integration.

We have earlier proved the Final Value Theorem of the degenerate Laplace transform (see Theorem (2.5), p.67 in [20]) as below

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\} \tag{2.19}$$

Taking the limit of (2.18) as $s \to 0$ gives

$$\lim_{s\to 0}\left[s\mathcal{L}_\lambda\left\{(1+\lambda t)^{-1} f(t)\right\}\right] = \log(1-\lambda) + C \tag{2.20}$$

A simple application of (2.19) in (2.20) along with the observation $\lim_{t\to\infty} f(t) = 0$ (as noted above) yields

$$\lim_{t\to\infty} f(t) = 0 = \log(1-\lambda) + C$$

which produces $C = -\log(1-\lambda)$. Substituting this value of C in (2.18) at once proves (2.9). We remark that the limiting case of (2.9) as $\lambda \to 0+$ produces the following known result (see, for instance, Problem 38, p.25 of [3])

$$\mathcal{L}\{\text{Ei}(t)\} = \frac{1}{s}\log(s+1) \tag{2.21}$$

Next we define the *degenerate error function* as follows:

Definition 2.5: The Degenerate Error Function: The *degenerate error function* is defined by the following integral

$$\text{erf}_\lambda(t) = \frac{2}{\sqrt{\pi}}\int_0^t e_\lambda^{-x^2}\,dx \tag{2.22}$$

The following theorem gives the degenerate Laplace transform of the *degenerate error function*.

Theorem 2.6: The Degenerate Laplace Transform of the Degenerate Error Function:

$$\mathcal{L}_\lambda\left\{\operatorname{erf}_\lambda(t)\right\}=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{\log(1+\lambda)}{\lambda}\right\}^n\frac{1}{(2n+1)s^{n+\frac{3}{2}}}\Gamma_{\frac{\lambda}{s}}\left(n+\frac{3}{2}\right) \tag{2.23}$$

$$\mathcal{L}_\lambda\left\{\operatorname{erf}_\lambda(t)\right\}=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left\{\frac{\log(1+\lambda)^{-2}}{\lambda}\right\}^n\frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{(s-\lambda)(s-2\lambda)\cdots(s-(2n+2)\lambda)} \tag{2.24}$$

Proof: Before beginning the proof of these results we mention that we interpret $e_\lambda^{-x^2}$ as follows

$$e_\lambda^{-x^2}=\left(e_\lambda^1\right)^{-x^2}=\left[(1+\lambda\cdot1)^{\frac{1}{\lambda}}\right]^{-x^2}=(1+\lambda)^{\frac{-x^2}{\lambda}}=\exp\left\{\frac{-x^2}{\lambda}\log(1+\lambda)\right\} \tag{2.25}$$

This leads us to

$$e_\lambda^{-x^2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{\log(1+\lambda)}{\lambda}\right\}^n x^{2n} \tag{2.26}$$

Hence we have

$$\operatorname{erf}_\lambda(t)=\frac{2}{\sqrt{\pi}}\int_0^t\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{\log(1+\lambda)}{\lambda}\right\}^n x^{2n}dx \tag{2.27}$$

Since the concerned series is uniformly convergent in the range of integration, we can interchange the operations of summation and integration in (2.27) which leads us to

$$\operatorname{erf}_\lambda(t)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{\log(1+\lambda)}{\lambda}\right\}^n\frac{t^{2n+1}}{(2n+1)} \tag{2.28}$$

Now taking the degenerate Laplace transform of both sides of (2.28) and utilizing the linearity property of the degenerate Laplace transform gives

$$\mathcal{L}_\lambda\left\{\operatorname{erf}_\lambda(t)\right\}=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{\log(1+\lambda)}{\lambda}\right\}^n\frac{\mathcal{L}_\lambda\left\{t^{2n+1}\right\}}{(2n+1)} \tag{2.29}$$

Kim and Kim [18, (3.19), p.246] have deduced that for $\alpha\in\mathbb{R}$ with $\alpha>-1$ and for $s>(\alpha+1)\lambda$,

$$\mathcal{L}_\lambda\left\{t^\alpha\right\}=\frac{1}{s^{\alpha+1}}\Gamma_{\frac{\lambda}{s}}(\alpha+1) \tag{2.30}$$

Now utilizing (2.30) on the right side of (2.29) generates (2.23) for $s>(2n+2)\lambda$.

Kim and Kim [18] have also shown that (see Theorem 3.2, p.246 in [18]) for $n\in\mathbb{N}, s>(n+1)\lambda$

$$\mathcal{L}_\lambda\left\{t^n\right\}=\frac{n!}{s^{n+1}}\frac{1}{\left(1-\frac{\lambda}{s}\right)\left(1-\frac{2\lambda}{s}\right)\cdots\left(1-\frac{(n+1)\lambda}{s}\right)} \tag{2.31}$$

An application of of (2.31) in (2.29) yields for $s>(2n+2)\lambda$

$$\mathcal{L}_\lambda\left\{\operatorname{erf}_\lambda(t)\right\}=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{\log(1+\lambda)}{\lambda}\right\}^n\frac{(2n+1)!}{(2n+1)s^{2n+2}}\frac{1}{\left(1-\frac{\lambda}{s}\right)\left(1-\frac{2\lambda}{s}\right)\cdots\left(1-\frac{(2n+2)\lambda}{s}\right)} \tag{2.32}$$

which on noting that

$$(2n+1)!=(2n+1)(2n)!=(2n+1)2^n n!1\cdot 3\cdot 5\cdots\cdots(2n-1)$$

generates (2.24) after a minor simplification.

Earlier we have defined the *degenerate Sine integral* (see Upadhyaya [21] Definition 2.2 and Theorem 2.3, p. 404) and the *degenerate Cosine integral* (see Upadhyaya [21] Definition 2.4 and Theorem 2.5, pp.405-406) and obtained their degenerate Laplace transforms. Now we venture into defining two another variants of the *degenerate Sine integral* and the *degenerate Cosine integral* and obtain their degenerate Laplace transforms. In order to distinguish the notation of these variants from their previously introduced analogs we use the sign of tilde $(\sim)$ over the new notations.

Definition 2.7: A New Variant of the Degenerate Sine Integral: We define a new variant of the *degenerate sine integral* by the following integral

$$\widetilde{\mathrm{Si}_\lambda(at)}=\int_0^t \frac{\sin_\lambda(au)}{u}du, \qquad a\in\mathbb{C} \tag{2.33}$$

It may be seen that in the limiting case $\lambda\to 0+$ and for $a=1$, (2.33) reduces to the classical *sine integral* given by (see, for instance, (8), Appendix C, p.255 [3])

$$\mathrm{Si}(t)=\int_0^t \frac{\sin u}{u}du \tag{2.34}$$

The next theorem gives the degenerate Laplace transform of the above new variant of the *degenerate sine integral*.

Theorem 2.8: Degenerate Laplace Transform of the New Variant of the Degenerate Sine Integral:

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Si}_\lambda(at)}\right\}=\frac{1}{s-\lambda}\tan^{-1}\left(\frac{a}{s-\lambda}\right) \tag{2.35}$$

Proof: Using (2.33) we can write

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Si}_\lambda(at)}\right\}=\mathcal{L}_\lambda\left\{\int_0^t \frac{\sin_\lambda(au)}{u}du\right\} \tag{2.36}$$

Put $u=tv$ on the right side of (2.36) which gives

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Si}_\lambda(at)}\right\}=\mathcal{L}_\lambda\left\{\int_0^1 \frac{\sin_\lambda(atv)}{v}dv\right\} \tag{2.37}$$

With the help of (1.16) this renders

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Si}_\lambda(at)}\right\}=\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}\left\{\int_0^1 \frac{\sin_\lambda(atv)}{v}dv\right\}dt \tag{2.38}$$

Since the variables of integration v and t in the above double integral are independent, we can interchange the order of integration in (2.38) to rewrite it as

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Si}_\lambda(at)}\right\}=\int_0^1 \frac{1}{v}\left\{\int_0^\infty (1+\lambda t)^{\frac{-s}{\lambda}}\sin_\lambda(atv)dt\right\}dv$$

Or,

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Si}_\lambda(at)}\right\}=\int_0^1 \frac{1}{v}\mathcal{L}_\lambda\left\{\sin_\lambda(atv)\right\}dv \tag{2.39}$$

Kim and Kim[18] have shown that (see (3.9), p.245)

$$\mathcal{L}_\lambda\left\{\sin_\lambda(at)\right\}=\frac{a}{(s-\lambda)^2+a^2} \tag{2.40}$$

Thus, (2.39), with the application of (2.40) yields

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Si}_\lambda(at)}\right\}=\int_0^1\frac{a}{\left[(s-\lambda)^2+a^2v^2\right]}dv=\frac{1}{s-\lambda}\left[\tan^{-1}\left(\frac{av}{s-\lambda}\right)\right]_{v=0}^{1}$$

which after a minor simplification produces (2.35). It may be seen that the result of (2.35) is only slightly different from the corresponding result of the Theorem (2.3), p. 404 of Upadhyaya [21]. Here it is also to be noted that for $a=1$ and $\lambda\to 0+$ the result of (2.35) reduces to the following known result (see Problem 36, p.24 [3])

$$\mathcal{L}_\lambda\left\{\mathrm{Si}(t)\right\}=\mathcal{L}_\lambda\left\{\int_0^t\frac{\sin u}{u}du\right\}=\frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right) \tag{2.41}$$

Definition 2.9: A New Variant of the Degenerate Cosine Integral: A new variant of the *degenerate Cosine integral* is defined as follows:

$$\widetilde{\mathrm{Ci}_\lambda(at)}=\int_t^\infty\frac{\cos_\lambda(au)}{u}du,\qquad a\in\mathbb{C} \tag{2.42}$$

which in the limiting case $\lambda\to 0+$ and for $a=1$ produces the *classical cosine integral* (see, for instance, (9), Appendix C, p.255 [3])

$$\mathcal{L}\left\{\mathrm{Ci}(t)\right\}=\int_t^\infty\frac{\cos u}{u}du \tag{2.43}$$

The next theorem gives the degenerate Laplace transform of the above new variant of the *degenerate Cosine integral*:

Theorem 2.10: Degenerate Laplace Transform of the New Variant of the Degenerate Cosine Integral:

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Ci}_\lambda(at)}\right\}=\frac{1}{2(s-\lambda)}\log\left(\frac{(s-\lambda)^2+a^2}{a^2}\right) \tag{2.44}$$

Proof: Taking the degenerate Laplace transform of both the sides of (2.42) yields

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Ci}_\lambda(at)}\right\}=\mathcal{L}_\lambda\left\{\int_t^\infty\frac{\cos_\lambda(au)}{u}du\right\} \tag{2.45}$$

Substituting $u=tv$ on the right side of (2.45) lends

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Ci}_\lambda(at)}\right\}=\mathcal{L}_\lambda\left\{\int_1^\infty\frac{\cos_\lambda(atv)}{v}dv\right\} \tag{2.46}$$

On using (1.16) in (2.46) we arrive at

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Ci}_\lambda(at)}\right\}=\int_0^\infty(1+\lambda t)^{\frac{-s}{\lambda}}\left\{\int_1^\infty\frac{\cos_\lambda(atv)}{v}dv\right\}dt \tag{2.47}$$

Since the variables of integration v and t in the above double integral are independent, thus we can change the order of integration in (2.47) to rewrite it as

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Ci}_\lambda(at)}\right\}=\int_1^\infty\left\{\int_0^\infty(1+\lambda t)^{\frac{-s}{\lambda}}\cos_\lambda(atv)dt\right\}\frac{1}{v}dv=\int_1^\infty\mathcal{L}_\lambda\left\{\cos_\lambda(atv)\right\}\frac{1}{v}dv \tag{2.48}$$

Kim and Kim [18] have proved that (see (3.8), p. 244 in [18])

$$\mathcal{L}_\lambda\left\{\cos_\lambda(at)\right\}=\frac{s-\lambda}{(s-\lambda)^2+a^2} \tag{2.49}$$

whose application in (2.48) lends

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Ci}_\lambda(at)}\right\}=\int_1^\infty\frac{(s-\lambda)}{v\left[(s-\lambda)^2+a^2v^2\right]}dv \tag{2.50}$$

The change of variable of integration $v^2=w$ in (2.50) leads to

$$\mathcal{L}_\lambda\left\{\widetilde{\mathrm{Ci}_\lambda(at)}\right\}=\int_1^\infty\frac{(s-\lambda)/a^2}{2w\left[\left(\frac{s-\lambda}{a}\right)^2+w\right]}dw=\frac{1}{2(s-\lambda)}\int_1^\infty\left[\frac{1}{w}-\frac{1}{\left(\frac{s-\lambda}{a}\right)^2+w}\right]dw \tag{2.51}$$

which on integration ultimately produces the result of (2.44). It is not out of context to remark here that in the limit $\lambda\to 0+$, for $a=1$ from (2.44) we can have the known result (see, for example, Problem 37, p.25 [3])

$$\mathcal{L}\left\{\mathrm{Ci}(t)\right\}=\frac{1}{2s}\log\left(s^2+1\right) \tag{2.52}$$

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