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Generalized Köthe-Toeplitz Duals of Double Difference Sequence Spaces Defined by Orlicz Functions

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Abstract

In this paper we define some difference sequence spaces and its sub-spaces using an Orlicz function and find their generalized Köthe-Toeplitz duals (?-duals).

Key words: Difference sequence spaces, Orlicz function, Generalized Köthe-Toeplitz Duals

Introduction

Throughout this paper ?, λ_∞, λ_1, c_1 and c_0 denotes the spaces of all, bounded, absolutely summable, convergent and null sequences x = (x_k) with complex terms respectively. The notion of difference sequence space was introduced by Kizmaz [6], who studied the difference sequence spaces $\lambda_\infty(\Delta)$, c(Δ) and $c_0(\Delta)$. The notion was further generalized by Et. M. and Colak [10] by introducing the spaces $\lambda_\infty(\Delta^m)$, c(Δ^m) and $c_0(\Delta^m)$.

Let m be a non-negative integers then for Z a given sequence space, we have

$$Z(\Delta^m) = \{x = (x_k) \in ? : (\Delta^m_{x_k}) \in Z\}$$

where

$$\Delta^m_x = (\Delta^m x_k) = (\Delta^{m-1} x_k - (\Delta^{m-1} \Delta^{m-1} x_{k+1})) \text{ and } \Delta^0 x_k = (x_k) \text{ for all } k ?? N, \text{ which}$$

is equivalent to the following binomial representation

$$\Delta^m x_k = \sum_{j=0}^{m} (-1)^j x_{k+j}\,.$$

After then Et. M. and Esi. A. [11] introduced the spaces $\lambda_\infty(\Delta^m_v)$, c($\Delta^m_v$) and $c_0(\Delta^m_v)$.

Let v = (v_k) be any fixed sequence of non-zero complex number and m be a positive integer then for Z a given sequence space, we have

Z(Δ^m_v) = {x = (x_k) ? : ($\Delta^m v_k x_k$) ? Z}

where $\Delta^m_v x = (\Delta^m v_k x_k) = (\Delta^{m-1} v_k x_k ? \Delta^{m-1} v_{k+1} x_{k+1})$, for all k ? N and so that

$$\Delta^m_v x_k = \sum_{q=0}^{m} (-1)^j \binom{m}{j} v_{k+j} x_{k+j}.$$

Taking v_k = (1, 1, ...), we get the spaces $\lambda_\infty(\Delta^m)$, c(Δ^m) and $c_0(\Delta^m)$ introduced and studied by Et. and Colak [10].

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to be Δ_2 –condition for all values of u, if these exists a constant K>0, such that

$M(2u) < KM(u)$, where $u \geq 0$.

The Δ_2 –condition is equivalent to $M(u) \leq K\lambda M(u)$, for all values of u and for $\lambda > 1$.

An Orlicz function M can always be represented in the following integral form :

$$M(x) = \int_0^x p(t)dt$$

where p, known as Kernal of M, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Consider the Kernel p(t) associated with the Orlicz function M(t), and let

$Q(s) = \sup\{t : p(t) \leq s\}$

The q possesses the same properties as the function p. Suppose now

$$N(x) = \int_0^x q(s)ds.$$

Then N is an Orlicz function. The functions M and N are called mutually complementary Orlicz functions.

Now we state the following results which can be found in [8].

Let M and N are mutually complementary Orlicz functions. Then we have (Young's inequality)

(i) For $x, y \geq 0$, $xy \leq M(x) + N(y)$

Also, we have

(ii) $M(\lambda x) < \lambda M(x)$ for all $x \geq 0$ and λ with $0 < \lambda < 1$.

Lindestrauss and Tzafriri [8] used the Orlicz function and introduced the sequence space λ_M as follows :

$$\lambda_M = \{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0\}$$

They proved that λ_M is a Banach space normed by

$$||x_k|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}.$$

A norm $||.||$ on a vector space X is said to be equivalent to a norm $||.||_0$ on X if there are positive number A and B such that for all $x \in X$, we have

$A||x||_0 \leq ||x|| \leq B||x||_0$.

This concept is motivated be the fact that equivalent norm on X define the same topology for X.

An isomorphism of a normed space X onto a normed space Y is a bijective linear operator $T : X \to Y$ which preserves the norm, i.e. for all $x \in X$,

$||T_x|| = ||x||$. (Hence T is isometric)

X is then called isomorphic with Y, X and Y are called isomorphic normed spaces.

2. Definitions and Notations

Let m, n be a positive integer. The we can have the following sequence spaces for an Orlicz function M as

$$c_0^2(M, v, \lambda, \Delta_n^m) = \{x = (x_{ij}) : \lim_{i+j\to\infty} M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) = 0, \text{ for some } \rho > 0\}$$

$$c^2(M, v, \lambda, \Delta_n^m) = \{x = (x_{ij}) : \lim_{i+j\to\infty} M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}| - L_1}{\rho}\right) = 0, \text{ for some complex number L and } \rho > 0\},$$

$$\lambda_\infty^2(M, v, \lambda, \Delta_n^m) = \{x = (x_{ij}) : \sup_{i,j} M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) < \infty, \text{ for some } \rho > 0\},$$

where $\Delta_n^m v_{ij}x_{ij} = \Delta_{n-1}^{m-1} v_{ij}x_{ij} - \Delta^{m-1} v_{i+1,j+1}x_{i+1,j+1}$, $\Delta_n^m v_{ij}x_{ij} =$

$$\sum_{2\le i+j\le m+n}\sum (-1)^{i'+j'} v_{i+i',j+j'} x_{i+i',j+j'}$$

It is obvious that

$$c_0^2(M, v, \lambda, \Delta_n^m) \subset c^2(M, v, \lambda, \Delta^m) \subset \lambda_\infty^2(M, v, \lambda, \Delta_n^m). \qquad (2.1)$$

Several authors have studied different algebraic and topological properties of such spaces. In this paper our main aim is to determine generalized Köthe-Toeplitz and Köthe-Toeplitz duals of such spaces.

Throughout the paper X will denote one of the sequence spaces c_0, c and λ_∞. The sequence spaces X(M, v, λ, Δ_n^m) are Banach spaces normed by

$$\|x\|_{\Delta_n^m} = \sum_{2\le k+\lambda\le m+n}\sum |v_{k\lambda}x_{k\lambda}| + \inf\{\rho > 0 : \sup_{i,j} M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) \le 1\}L \qquad (2.2)$$

Now, we take

$$\Delta_n^m v_{ij}\lambda_{ij}x_{ij} = \sum_{2\le i+j\le m+n}\sum (-1)^{i'+j'} \binom{m}{i'}\binom{n}{j'} v_{m-i',n-j'}\lambda_{i-i',j-j'} x_{i-i',j-j'}$$

It is trivial ($\Delta_n^m v_{ij}x_{ij}$) ∈ X(m) if anf only if ($\Delta_n^m v_{ij}x_{ij}$) ∈ X(M). Now for x ∈ X(M, v, Δ_n^m), we define

$$\|x\|_{\Delta_n^{(m)}} = \inf\{\rho > 0 : \sup_{i,j} M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) \le 1\}.$$

It can be shown that X(M, v, λ, Δ_n^m) is a BK-space under the norm $\|x\|_{\Delta_n^m}$ and the norms $\|x\|_{\Delta_n^m}$ and $\|x\|_{\Delta_n^m}$ are equivalent. Obviously Δ_n^m : X(M, v, λ, $\Delta^{(m)}$) → X(M) denoted by $\Delta_{(n)}^{(m)}(x) = y = (\Delta_n^m v_{ij}\lambda_{ij}x_{ij})$, is isometric isomorphism. Hence c_0^2(M, v, λ, Δ_n^m), c^2(M, v, λ, Δ_n^m) and λ_∞^2(M) are isometrically isomorphic to c_0^2(M), c^2(M) and λ_∞^2(M) respectively. From abstract point of view X(M, v, λ, Δ_n^m) is identical to X(M).

Now we define the spaces $\bar{c}_0^2$(M, v, λ, Δ_n^m) is subspace of c_0^2 (M, v, λ, Δ_n^m) consisting of those x in c_0^2(M, v, λ, Δ_n^m) such that $\lim_{i+j\to\infty} M\frac{\left(|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|\right)}{d} = 0$ for each d > 0.

Similarly we define $\bar{c}^2(M, v, \lambda, \Delta_n^m)$ and $\bar{\lambda}_\infty^2(M, v, \lambda, \Delta_n^m)$ as subspace of $\bar{c}^2(M, v, \lambda, \Delta_n^m)$ and $\bar{\lambda}_\infty^2(M, v, \lambda, \Delta_n^m)$ respectively. The topology of $\bar{X}(M, v, \lambda, \Delta_n^m)$ is the one it inherits from $\|\cdot\|_{\Delta_n^m}$.

It is obvious that

$\bar{c}_0^2(M, v, \lambda, \Delta_n^m) \subset \bar{c}^2(M, v, \lambda, \Delta_n^m) \subset \bar{\lambda}_\infty^2(m, v, \lambda, \Delta_n^m)$.

Also as above we can show that $\bar{X}(M, v, \lambda, \Delta_n^m)$ are isometrically isomorphic to $\bar{X}(M)$.

Moreover $X^2(M, v, \lambda, \Delta_n^m) \subset X^2(M, v, \lambda, \Delta_{n+1}^{m+1})$ and $\bar{X}^2(M, v, \lambda, \Delta_n^m) \subset X^2(M, v, \lambda, \Delta_{n+1}^{m+1})$ which can be shown by repeated application of the following inequality

$$M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|}{2\rho}\right) \leq \frac{1}{2}M\left(\frac{|\Delta_{n-1}^{m-1} v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) + \frac{M}{2}\left(\frac{|\Delta_{n-1}^{m-1} v_{i+1,j+1}\lambda_{i+1,j+1}x_{i+1,j+1}|}{\rho}\right).$$

3. Generalized Köthe-Toeplitz Duals

In this section our main aim is to determine ⍰-dual and ⍰-dual of the sequence spaces $c_0^2(M, v, \lambda, \Delta_n^m)$, $c^2(M, v, \lambda, \Delta_n^m)$, $\lambda_\infty^2(M, v, \lambda, \Delta_n^m)$, $\bar{c}_0^2(M, v, \lambda, \Delta_n^m)$, $\bar{c}^2(M, v, \lambda, \Delta^m)$ and $\bar{\lambda}_\infty^2(M, v, \lambda, \Delta_n^m)$.

Definition (3.1). Let E be a sequence space and r ≥ 1. Then the ⍰-dual of E is defined as

$$E^{⍰} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k r_k|^r < \infty, \text{ for all } x = (x_k) ⍰ E\}.$$

If we take r = 1, then we have Köthe-Toeplitz duals of E, i.e.,

$$E^{⍰} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k r_k| < \infty, \text{ for all } x = (x_k) ⍰⍰E\}.$$

If $E \subset F$, then $F^z \subset E^z$ for z = ⍰, ⍰.

Lemma (3.2). ([3]). Let m be a positive integer. Then there exists positive constants c_1 and c_2 such that

$$c_1(i+j)^{m+n} \leq \binom{m+i}{i}\binom{m+j}{j} \leq c_2(i+j)^{m+n}, \qquad i+j = 0, 1, 2, \ldots$$

Lemma (3.3). $x \in \lambda_\infty^2(M, v, \lambda, \Delta_n^m)$ implies that $\sup_{i,j} M\left(\frac{|(i+j)^{-1}\Delta_{n-1}^{m-1} v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) < \infty$, for some $\rho >$ 0.

Proof. Let $x \in \lambda_\infty^2(M, v, \lambda, \Delta_n^m)$, then

$$\sup_{i,j} M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) < \infty, \text{ for some } \rho > 0.$$

Then there exists a U > 0, such that

$$M\left(\frac{|\Delta_n^m v_{ij}\lambda_{ij}x_{ij}|}{\rho}\right) < U, \text{ for all } i,j \in N.$$

Taking ⍰ = (i+j)⍰, i,j > 1 being fixed number, we have

$$M\left(\frac{|\Delta_{n-1}^{m-1} v_{11}\lambda_{11}x_{11} - \Delta_{n-1}^{m-1} v_{i+1,j+1}\lambda_{i+1,j+1}x_{i+1,j+1}|}{\eta}\right) = M\left(\frac{\left|\sum\sum_{2\le i'+j'\le i+j} \Delta_n^m v_{i'j'}\lambda_{i'j'}x_{i'j'}\right|}{(i+j)\rho}\right)$$

$$\le M\left(\frac{\left|\sum\sum_{2\le i'+j'\le i+j} \Delta_n^m v_{i'j'}\lambda_{i'j'}x_{i'j'}\right|}{(i+j)\rho}\right)$$

$$\le \frac{1}{i+j} M\left\{\frac{|\Delta_n^m V_{11}\lambda_{11}x_{11}|}{\rho} + \frac{|\Delta_n^m V_{22}\lambda_{22}x_{22}|}{\rho} + \ldots + \frac{|\Delta_n^m V_{ij}\lambda_{ij}x_{ij}|}{\rho}\right\}$$

$$\le \frac{1}{i+j}(U + U + \ldots + U) = O(i,j)$$

Now the result follows from the following inequality using the convexity of M

$$|\Delta_{n-1}^{m-1} v_{i+1,j+1}\lambda_{i+1,j+1}x_{i+1,j+1}| \le |\Delta_{n-1}^{m-1} v_{11}\lambda_{11}x_{11}|$$
$$+ |\Delta_{n-1}^{m-1} v_{11}\lambda_{11}x_{11} - \Delta_{n-1}^{m-1} v_{i+1,j+1}\lambda_{i+1,j+1}x_{i+1,j+1}|.$$

Lemma (3.4). $\sup_{i,j} M\frac{(|(i+j)^{-(m+n)}\Delta v_{ij}\lambda_{ij}x_{ij}|)}{\rho} < \infty$ implies $\sup_{i,j} M\frac{(|(i+j)^{-(m+n+1)}\Delta v_{ij}\lambda_{ij}x_{ij}|)}{\rho} < \infty$, for all n □ N and some □ > 0.

Proof. Proof follows from Lemma (3.3).

Lemma (3.5). (i) $\sup_{i,j} M\frac{|(i+j)^{-1}\Delta_{n-1}^{m-1} v_{ij}\lambda_{ij}x_{ij}|}{\rho} < \infty$ implies $\sup_{i,j} M\frac{(|(i+j)^{-(m+n)}\Delta v_{ij}\lambda_{ij}x_{ij}|)}{\rho} < \infty$ for some □ > 0.

(iii) x □□ λ_∞^2 (□□□v□□□,□ Δ_n^m) implies $\sup_{i,j} |(i+j)^{-(m+n)}\Delta_{n-1}^{m-1} v_{ij}\lambda_{ij}x_{ij}| < \infty$.

Proof. (i) Proof follows from Lemma (3.4).

(ii) Combining the Lemma (3.4) and part (i).

(iii) Proof follows from part (i).

Remark 1. Similar results as in Lemma (3.5) hold for $\bar{\lambda}_\infty^2$ (M, v, □, Δ_n^m) also, where the statement "for some □ > 0".

Theorem (3.6). Let M be an Orlicz function. Then

(i) [c_0^2(M, v, □, Δ_n^m)]=[c^2(M, v, □, Δ_n^m)]□=[λ_∞^2(M, v, □, Δ_n^m)]=u_1 (3.1)

(ii) ${}^2u_1^\eta = {}^2u_2$. (3.2)

where u_1^2 = {a = (a_{ij}) : $\sum\sum_{2\le i+j\le\infty}$ $|(i+j)^{m+n} v_{ij}□_{ij}a_{ij}|^{r+s} < \infty$ }

$$u_2^2 = \{a = (a_{ij}) : \sup_{i,j} |(i+j)^{\square(m+n)} v_{ij} \square_{ij} a_{ij}|^{r+s} < \infty\}$$

Proof. (i) First we suppose that a $\square$ u_1^2, then

$$\sum\sum_{2 \le i+j \le \infty} |(i+j)^{(m+n)} v_{ij} \square_{ij} a_{ij}|^{r+s} < \infty .$$

Let x $\square$ λ_∞^2 (M, v, $\square$, Δ_n^m)

$$\sum\sum_{2 \le i+j \le \infty} |a_{ij} \square_{ij} x_{ij}|^{r+s} = \sum\sum_{2 \le i+j \le \infty} |(i+j)^{(m+n)} (v_{ij})^{\square 1} \square_{ij}\, a_{ij}|^{r+s}\, |(i+j)^{\square(m+n)} v_{ij} x_{ij}|^{r+s}$$

$$\le \sum\sum_{2 \le i+j \le \infty} |(i+j)^{(m+n)} (v_{ij})^{\square\square} a_{ij}|^{r+s} < \infty, \text{ for each } x \square\square \lambda_\infty^2 (M, v, \square, \Delta_n^m)$$

by using Lemma (3,5) (iii). Thus we have to show that

$$u_1^2 \subset [\lambda_\infty^2 (M, v, \square, \Delta_n^m)]^\square \tag{3.3}$$

Conversely let a $\square\square$ u_1^2. Then for each i,j, we have

$$\sum\sum_{2 \le i+j \le \infty} |(i+j)^{m+n} (v_{ij})^{\square 1} \square_{ij} a_{ij}|^{r+s} = \infty .$$

So we can find a sequence (n_i) of positive integer n_i with $n_1 < n_2 < \ldots..$, such that

$$\sum_{i+j \ge n_i + 1} |(i+j)^{m+n} (v_{ij})^{\square 1} \square_{ij} a_{ij}|^{r+s} > (i)^{r+s}$$

Now we define a sequence $x = (x_{ij})$ as

$$x_{ij} = \begin{cases} 0 & (1 \le i+j \le n_1) \\ \dfrac{(i+j)^{(m+n)} \lambda_{ij} (v_{ij})^{-1}}{i} & (ni+1 < i+j \le n_{i+1},\ i = 1, 2, 3 \ldots.) \end{cases}$$

Then it is easy to see that $x = (x_{ij})$ $\square$ c_0^2 (M, v, $\square$, Δ_n^m). But

$$\sum\sum_{2 \le i+j \le \infty} |a_{ij} \lambda_{ij} x_{ij}|^{r+s} = \sum_{i=1}^{\infty} \{ \sum_{i+j \ge n_i+1} |a_{ij} \lambda_{ij} x_{ij}|^{r+s} \} > \sum_{i=1}^{\infty} 1 = \infty .$$

Which contradicts that a $\square$ [c_0^2 (M, v, $\square$, Δ_n^m)]$^\square$. Hence

$$[c_0^2 (M, v, \square, \Delta_n^m)]^\square \subset u_1^2 . \tag{3.4}$$

Since c_0^2 (M, v, $\square$, Δ_n^m)]$^\square \subset c^2$(M, v, $\square$, Δ_n^m) $\subset \lambda_\infty^2$ (M, v, $\square$, Δ_n^m) implies [λ_∞^2 (M, v, $\square$, Δ_n^m)]$^\square \subset$ [c^2(M, v, $\square$, Δ_n^m)]$^\square \subset$ [c_0^2 (M, v, $\square$, Δ_n^m)]$^\square$, (3.1) follows from (3.3) and (3.4).

(ii) Proof is similar to proof of part (i).

Theorem (3.7). Let M be an Orlicz function. Then

(i) [$\bar{c}_0^2$ (M, v, $\square$, Δ_n^m)]$^\square$ = [c^2(M, v, $\square$, Δ_n^m)]$^\square$ = [λ_∞^2 (M, v, $\square$, Δ_n^m)] = u_1^2

(ii) ${}^2u_1^n = {}^2u_2$,

where $u_1^2 = \{a = (a_{ij}) : \sum\sum_{2 \le i+j \le \infty} |(i+j)^{m+n} (v_{ij})^{\square 1} \square_{ij} a_{ij}|^{r+s} < \infty\}$,

$u_2^2 = \{a = (a_{ij}) : \sup_{i,j} |(i+j)\square(m+n) v_{ij} \square_{ij} a_{ij}| < \infty\}$.

Proof. The proof is similar to that of theorem (3.6).

If we take $v_{ij} = \begin{vmatrix} 1,1,\ ....1 \\ 1,1,\ ....1 \\ 1,1,\ ....1 \\ .... \\ 1,1,\ ....1 \end{vmatrix}$ I theorem (3.6) and Theorem (3.7). Then we obtain the following Corollary :

Corollary (3.8). For X = c_0^2, c^2 and λ_∞^2 .

(i) $[X^2(M, □, \Delta_n^m)]^{□} = [X(M, □, \Delta_n^m)]^{□} = H_1^2$

(ii) $^2H_1^\eta = {}^2H_1$

where $H_1^2 = \{a = (a_{ij}) : \sum\sum_{2 \le i+j \le \infty} |(i+j)^{m+n} □_{ij} a_{ij}|^{r+s} < \infty\}$

$H_2^2 = \{a = (a_{ij}) : \sup_{i,j} |(i+j)^{□(m+n)} □_{ij} a_{ij}| < \infty\}$

If we take $v_{ij} = \begin{vmatrix} 1,1,\ ....1 \\ 1,1,\ ....1 \\ 1,1,\ ....1 \\ .... \\ 1,1,\ ....1 \end{vmatrix}$ and m,n = 0 in the theorem (3.6) and Theorem (3.7), then we obtain the following corollary :

Corollary (3.9). For X = c_0^2, c^2 and λ_∞^2

(i) $[X^2(M)]^{□} = [\overline{X}^2(M)]^{□} = M_1$

(ii) $^2M_1^\eta = {}^2M_2$

where $M_1^2 = \{a = (a_{ij}) : \sum\sum_{2 \le i+j \le \infty} |□_{ij} a_{ij}|^{r+s} < \infty\}$

$M_2^2 = \{a = (a_{ij}) : \sup_{i,j} |□_{ij} a_{ij}| < \infty\}$.

Theorem (3.10). Let M be an Orlicz function. Then

(i) $[c_0^2(M, v, □, \Delta_n^m)]^{□} = [c^2(M, v, □, \Delta_n^m)]^{□} = [\lambda_\infty^2(M, v, □, \Delta_n^m)] = D_1^2$, (3.5)

(ii) $^2D_1^\alpha = {}^2D_2$ (3.6)

where $D_1 = \{a = (a_{ij}) : \sum\sum_{2 \le i+j \le \infty} |(i+j)^{□(m+n)} (v_{ij})^{□1} □_{ij} a_{ij}| < \infty\}$

$D_2 = \{a = (a_{ij}) : \sup_{i,j} |(i+j)^{m+n} v_{ij} □_{ij} a_{ij}| < \infty\}$

Proof. (i) First we suppose that a □□ D_1^2, then

$\sum\sum_{2 \le i+j \le \infty} |(i+j)^{□(m+n)} (v_{ij}^{□})^{□1} □_{ij} a_{ij}| < \infty$.

Let x □ $\lambda_\infty^2(M, v, □, \Delta_n^m)$. Then

$\sum\sum_{2 \le i+j \le \infty} |a_{ij} □_{ij} x_{ij}| = \sum\sum_{2 \le i+j \le \infty} |(i+j)^{□(m+n)} v_{ij} □_{ij} a_{ij}|\ |(i+j)^{□(m+n)} (v_{ij})^{□1} □_{ij} a_{ij}|$

$\le \sum\sum_{2 \le i+j \le \infty} (i+j)^{m+n} |(v_{ij})^{□1} □_{ij} a_{ij}| < \infty$

for each x □ $\lambda_\infty^2(M, v, □, \Delta_n^m)$, be Lemma (3.5) (iii). Thus we have to show that

$$D_1^2 \subset [\lambda_\infty^2 (M, v, ?, \Delta_n^m)]^{?}. \qquad (3.7)$$

Conversely let a ? D_1^2. Then for some i,j, we have

$$\sum\sum_{2\le i+j\le\infty} |(i+j)^{?(m+n)}(v_{ij}^{?})^{?1}?_{ij}a_{ij}| = \infty.$$

So, we can find a sequence (n_i) of positive integer n_i with $n_1<n_2<\ldots$, such that

$$\sum\sum_{i+j\ge n_i+1} |(i+j)^{m+n}(v_{ij})^{?1}?_{ij}a_{ij}| > i.$$

Now we define a sequence x = (x_{ij}) as

$$x_{ij} = \begin{cases} 0 & 1\le i+j\le n \\ \dfrac{v_{ij}^{-1}(i+j)^{m+n}}{i}, & (n_i+1 < i+j \le n_i+1 : i = 1, 2, \ldots) \end{cases}$$

Then it is easy to verify that x = (x_{ij}) ? (M, v, ?, Δ_n^m). But

$$\sum\sum_{2\le i+j\le\infty} |a_{ij}?_{ij}x_{ij}| = \infty,$$

which contradicts that a ? $[c_0^2 (M, v, ?, \Delta_n^m)]$. Hence, we have

$$[c_0^2 (M, v, ?, \Delta_n^m)] \subset D_1^2. \qquad (3.8)$$

Since $[\lambda_\infty^2 (M, v, ?, \Delta_n^m)] \subset [c^2(M, v, I, \Delta_n^m)] \subset [c_0^2 (M, v, ?, \Delta_n^m)]^{?}$, so (3.5) follows from (3.7) and (3.8).

(ii) Proof is similar to proof of part (i).

Theorem (3.11). Let Me be an Orlicz function. Then

(i) $[\bar{c}_0^2 (M, v, ?, \lambda_\infty^2)]^{??} = [\bar{c}^2 (M, v, ?, \Delta_n^m)]^{??} = [\bar{\lambda}_\infty^2 (M, v, ?, \Delta_n^m)]^{??} = D_1^2$,

(ii) ${}^2D_1^{\alpha} = {}^2D_2$

where $D_1^2 = \{a = (a_{ij}) : \sum\sum_{2\le i+j\le\infty} |(i+j)^{m+n}(v_{ij})^{?1}?_{ij}a_{ij}| < \infty\}$

$D_2 = \{a = (a_{ij}) : \sup_{i,j} |(i+j)^{?(m+n)}v_{ij}a_{ij}| < \infty\}$.

Proof. The proof is similar to that of theorem (3.10).

If we take $v_{ij} = \begin{vmatrix} 1, 1, \ldots\ldots\ldots 1 \\ \ldots\ldots\ldots\ldots\ldots \\ 1, 1, \ldots\ldots\ldots 1 \end{vmatrix}$ and m,n = 0 in theorem (3.10) and theorem (3.11), then we obtain following corollary :

Corollary (3.12). For X = c_0^2, c^2 and λ_∞^2

(i) $[X^2(M)]^{?} = [\overline{X}^2 (M)]^{?} = G_1$

(ii) ${}^2G_1^{\alpha} = {}^2G_2$

where $G_1^2 = \{a = (a_{ij}) : \sum\sum_{2\le i+j\le\infty} |?_{ij}a_{ij}| < \infty\} = \lambda_1^2$,

$G_2^2 = \{a = (a_{ij}) : \sup_{i,j} |?_{ij}a_{ij}| < \infty\} = \lambda_\infty^2$.

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