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A New Approach to Find the Multi- Fractal Dimension ofMulti-Fuzzy FractalAttractor Sets Based on the IteratedFunction System

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Abstract.

In nature, objects are not single fractal sets but are a collection of complex multiple fractals that characterise the multifractal space, a generalisation of fractal space. While fractal space includes a fractal set, a multi-fractal space includes the union of fractals. A fuzzy fractal space is a fuzzy metric space and is an approach for the construction, analysis, and approximation of sets and images that exhibit fractal characteristics. The finite Cartesian product of fuzzy fractal spaces is called the multi-fuzzy fractal space. We propose in this paper, a theoretical proof to define the multi-fractal dimensions FD of a multi-fuzzy fractal attractor of n objects for the self-similar fractals sets A $= \prod_{i=1}^{n} A_i = (A_1, A_2, \ldots A_n)$ of the contraction mapping W∗∗ :$\prod_{i=1}^{n} H(F(X_i)) \rightarrow \prod_{i=1}^{n}$ =1 H(F(Xi))with contractivity factor r=max (r_i,=1,2….) where H(F(x_i) is a fuzzy fractal space for each i =1,2…..n;overacompletemetricspace$\prod_{i=1}^{n} H(F(X_i), D*)$ then for all B_i that belong to H(F(X_i)), there exists B^* belongs to $\prod_{i=1}^{n} H(F(X_i))$ such that $W^{**}\left(B^* = \prod_{i=1}^{n} B_i\right) = \prod_{i=1}^{n}\left(\cup_{j=1}^{n} \cup_{k=1}^{k(i,j)} \omega_{ij}^{*k}(B_j) = \prod_{i=1}^{n} W_i(B^*)\right)$ By supposing that $M(t) = \left(\sum_k \left(r_{ij}^{*k}\right)^{FD}\right)_{n\times n}$ is the matrix associated with the the contraction mapping $\omega^{*k}_{ij}$with contraction factor r^{*k}_{ij},∀i, j = 1,2,...,n,∀k = 1,2,...,k(i, j), for all t ≥ 0,and h(t) = det(M(t)−I) . Then, we prove that if there exists a FDsuch that; h(FD) = 0, then FDis the multi fractal dimension for the multi fuzzy-fractal sets of IFS; and M(FD) has a fixed point in R^n.

Keywords: Fractal space, multi-fuzzy fractal space, IFS, box-counting dimension, fractal dimension. PACS: 05.45.Df, 03.00.00

INTRODUCTION

Mandelbrot [1] developed the geometrical idea that exceeded classical geometry. He reluctantly defined the term fractal, and he illustrated that dimension, a real number and not an integer, is a measure of complexity, and he defined fractal as the set for which the Hausdorff dimension robustly transcends the topological dimension . There are now several ways to describe a fractal. Barnsley [2] singled out the iterated function system (IFS) method as the fractal set is comprised of parts, each of which is a shrunken copy of the whole set. While a fractal refers to a set or an object, a multi-fractal refers to a union of sets of many objects with different properties [3]. The multi-

fractal was first studied by physicists in the 1980s and has since been studied by many authors. Al-Shameri in 2001 [4] dealt with the space of multi-fractals as well as the generalisation of IFS to the multi-fractal space.
Fuzzy sets attracted considerable interest by Chang in 1968 [5] used the fuzzy set theory to define and introduce fuzzy topological space. Fadel in 1998 [6] studied fuzzy sets to prove the fuzzy fixed point theorem. Al-Saidi in 2003 [7] proved a generalisation to the fuzzy metric space to build a modern space to represent the finite Cartesian product of fuzzy fractal spaces called the multi-fuzzy fractal space.
Many types of fractals dimensions exist, such as the capacity dimension and box dimension. Ahmed and Qaeed [8] calculate the dimension D of a multi-fractal for n = 3and4 objects of the self-similar fractal sets . In this paper, we prove, that there exist a multi-fractal dimension of multi-fuzzy fractals constructed by similarity contraction mapping in Euclidean space.
The paper is arranged as follows. Section 2 covers the theoretical backgrounds of fractals , multi-fractals and multi-fuzzy fractal space with several types of fractal dimensions represented.
Since no solution exists for the general polynomial of degree $n \geq 5$, the existence of a multi- dimension for a multifuzzyfractalattractorconstructedbyfuzzycontractionsimilaritiesmappingof nobjectsispresentedinSection 3 for a special condition. Conclusion is presented in Section 4.

Theoretical background

This section provides a condensed mathematical background of the notions, definitions, and theorems of multi-fractal space, multi-fuzzy fractal space, and fractal dimension. A more detailed description of these topics is provided in [2,9,10,11].
Let (X,d) be a metric space and let H(X) be the set of all non-empty compact subsets of X, which is called the space of fractals. The Hausdorff metric on H(X) is defined as: for all A, B ∈ H(X),h(A, B) = max {maxa∈Aminb∈B d(a,b),maxb∈B mina∈A d(b,a)}. Then, (H(X),h) is a complete metric space if (X,d) is also complete.
The contraction mapping ωi for contractivity r, where $0 \leq r < 1$, in a metric space (x,d) is defined by
$d(\omega(x),\omega_i(y)) \leq r\, d(x,y)$, $\forall_x,y \in X$, where ω is the similarity mapping if there is a number $r \geq 0$, where $d(\omega(x),\omega(y)) = r\, d(x,y)$, ∀x,y ∈ X for all x,y ∈ X.
IFS consists of a complete metric space (X,d) together with a finite set of the contraction mapping ωi for contractivity factor r = max{ri,i = 1,2,,n}. Let W: H (x) → H (x) be defined by W n for all B ∈ H (x). Then, W is a contraction mapping on H (X) with contractivity r = max{rn: n = 1,2,...n} on the complete metric space (H (x),h). By the fixed point theorem, there exists a unique element A ∈ H(X), such that W (A) = A and limn→∞ Wk(B) = A for all B ∈ H(X) [2]. The set A is called the fractal attractor of IFS. When A is a disjoint union or just touching then it is referred to as self -similarity.

Let (F(X), D∗) be the fuzzy metric space, and if (X;ω1,ω2,...,ωn) is a IFS, then (,...,ω∗n) is a fuzzy IFS, where (F(X), D∗) is a complete metric space, and ω∗i : F(X) → F(X), is a contraction mapping for contractivtiy factor ri ,for i = 1,2,,n.[6,12].
The collection of all non- empty compact subsets of F(X) is called the fuzzy fractal space and denoted byH(F(X)). Also, when (X,d) is a complete metric space, then (H(F(X)),h∗) is also a complete metric space, and W∗ : H(F(X)) → H(F(X)), is a contraction mapping for contractivity factor r = max{r1,r2,...,rn}. By the fixed point theorem on the fuzzy fractal space, there exists a unique fixed fuzzy subset A∗ of H(F(X)), such that W∗(A∗) = A∗. The multi-fractals, can be considered an object or a point in the product of space of H(Xi), denotes the multi-fractal space, defined by: H∗(X) = H (X1) × H (X2)×...× H (Xn) = ni=1 H(Xi). Let (Xi,di) be a complete metric space, then there exists a corresponding fractal space (H(Xi),hi), and contraction mapping Wi : H(Xi) → H(X). The contraction mapping W∗∗ : H∗(X) → H∗(X) on the product fractal space H∗(X) for contractivity factor r{r1,r2,...,rn} ,is defined by :W∗∗(B^*) = (W_1(B^*),W_2(B^*),...,W_n(B^*))=$\prod_{i=1}^{n} W(Bi))$ where B∗ = (B1, B2,..., Bn), which has a unique non empty compact set $A^*=\prod_{i=1}^{n}$ Ai; and $\prod_{i=1}^{n} Xi$[4,6].
Now, let (F(Xi), Di) be a complete fuzzy space [7], and H(F(Xi)) is a fuzzy fractal space for each i = 1,2,...,n, . A multi-fuzzy fractal space H∗∗(X) is defined by H∗∗ = n), and the metric function D∗ is defined as: D∗: H∗∗ (X) × H∗∗ (X) → R, D∗ (A∗, B∗) = max {Di (Ai, Bi)},where A∗ = (A1, A2, ..., An),B ∗ = (B1, B2, ...,Bn) ∈H∗∗(X). Then (H∗(X),D∗) is a complete metric space. Let (H∗(X), D∗) be a complete metric space, and W∗i : i = 1,2,...,n be a contraction mapping on H(F(Xi)) with contractivity factor ri then the contraction mapping W∗∗∗ : n i=1 H(F(Xi))→n i=1 H(F(Xi)), with contractivity factor r = max{ri,i = 1,2...n} is denoted as :W∗∗∗(B = n i=1 Bi) = n i=1 (n j=1k(i,j) k=1 ω∗kij(Bj) = n i=1 Wi(B)). Then by the fuzzy contraction mapping theorem,W∗∗∗ hasauniquepoint A∗ = A1,A2,...,An,suchthattheiterateofanyotherfuzzysetin H∗∗(X)converges to A∗, and A∗ is called the multi- fuzzy fractal attractor set on (H∗∗(X), D∗) [7,13]
Finally, we recall some definitions of the fractal dimension, which we required in our work.
Let A be a totally bounded subset of a metric space (X,d) and N(A,) be the minimum number of closed balls withradius>0thatcover A.Then,iflim→∞ lnN(A,) ln(1) exists,itiscalledthecapacitydimension.When (n) = rn for 0 < r < 1. Then D(A) = lim→∞ lnN(A) ln(1) = limn→∞ lnN(A,r) n ln(r) and is called the box dimension. Also, if A,B ∈ H(Rm) and A⊂ B, then DB(A)≤ DB(B). Also if DB(B) < DB(A), then DB (A∪B) = DB (A)[2,9].

Multi-dimensions of multi-fuzzy fractal attractor sets based on IFS

We present the idea of the multi- fractal dimension with theoretical proof there exists a box dimension to the multifuzzy fractal attractor sets based on IFS. In this paper we prove there exists a box multi-fractal dimension FD of the multi- fuzzy fractal attractor set $M(t)=\left(\sum_k \left(r_{ij}^{*k}\right)^{FD}\right)_{n\times n}$ for i = 1,2,...n, as an attractor of the fuzzy contraction mapping, W∗∗ , over the complete metric space (H∗∗(X), D∗)), by calculating FD as a maximum root of the functionh(t) = |M(t) − I| , when the matrix $\left(\sum_k \left(r_{ij}^{*k}\right)^{FD}\right)_{n\times n}$ has a non trivial fixed point in Rn. We consider all metricspaces(Xi,di) are equal, and n is a given positive integer. Then (Xn,d) is the product metric space of (Xi,di), where d(x,y) = max{di(xi,yi):i = 1,2,...,n}.

d(x,y) = max{di(xi,yi):i = 1,2,...,n}.

(H∗(X),h) is the product of the fractal spaces, (H(Xi),hi), for n times, where h(A,B) = max{hi(Ai,Bi):i = 1,2,...,n}.(H∗∗(X),D∗)) is the product space of the fuzzy fractal spaces, H(F(Xi)) such that D∗(A∗,B∗) = max 1≤i≤n (Di(Ai,Bi)). The multi- fuzzy fractal attractor set, A∗∗ = A1,A2,...,An , is union of non-overlapping or just touching components Ai ⊆ X. Then by theorem DB(A∪B) = DB(A) [2,9],enabling the following definition.

Definition1 Let A = (A1,A2,...An) ∈n i=1 H(F(Xi)) be multi-fuzzy fractal attractor set and (ω∗kij) is a partition (e.g., non-overlapping or just touching), then the box dimension of A is the maximum box dimension of Ai, ∀i = 1,2,...,n, i.e, the FD is = DB(A) = max{DB(Ai):i = 1,2,...,n}.Then, FD is called the box multi-box fractal dimension of the multi- fuzzy fractal attractor set A.

The following theorem transforms the entire notation from geometrical figures to matrix algebra, by using notions of the existence of an attractor, and the dimension of this attractor as a multi-fractal.

Theorem 1 Let (X, d) be a complete metric space, and n be a positive integer, and let$\left\{\prod_{i=1}^{n} H(X_i), \omega_{ij}^{k}\right\}$beIFS with contraction similarity r_{ij}k∀i, j = 1,2,...n, ∀ k = 1,2,...k(i, j) ,and the multi- fractal attractorA = (A1,A2,...An)∈$\prod_{i=1}^{n}$ H(Xi),hasmulti-fractaldimension FD.Thenthematrix$\left(\sum_k \left(r_{ij}^{k}\right)^{FD}\right)_{nxn}$ hasanon-trivial fixed pointC in the space Rn, such that C = $\left(\sum_k \left(r_{ij}^{k}\right)^{FD}\right)_{nxn}$.

Since there exists no solution for the general polynomial of degree n ≥ 5, we present a theorem to find the multi- box dimension for the special condition of the multi-fuzzy fractal set generated by similarities containing n objects, by using the definitions of an eigenvalue of the matrix M, the eigenvector associated with an eigenvalue is a characteristic polynomial of M, the spectra radius of M [14]and by using theorem of 1, using matrix algebra we introduce the multi-box dimension of n ≥ 5 objects.

Theorem2*Let* H∗∗(X) = $\prod_{i=1}^{n}$ H(F(Xi)) be a multi-fuzzy fractal space, and W∗∗ :$\prod_{i=1}^{n} H\left(F(X_i)\right)\rightarrow\prod_{i=1}^{n}$ =1 H(F(Xi)), be a contraction mapping for a contractivity similar factor r = max{ri : i = 1,2,...,n}, denoted as: $W^{**}\left(B^* = \prod_{i=1}^{n} B_i\right) = \prod_{i=1}^{n}\left(\cup_{j=1}^{n} \cup_{k=1}^{k(i,j)} \omega_{ij}^{*k}\left(B_j\right)\right) = \prod_{i=1}^{n} W_i^*\left(B^*\right)$ where i, j = 1,2,...,n . Let M(t) = $(t) = \left(\sum_k (r_{ij}^{*k}\right)$ be the matrix associated with IFS, for all t ≥ 0 . If there exists an FD such that the sum of allentries of each column of the matrix M(FD)−I is zero, then FD is the multi- box fractal dimension of the multi-fuzzy fractal attractors in the complete metric space (H∗∗(X),D∗).

Proof1 LetA(t) = M(t)−1 for all t ≥ 0 , and let |h(t)| = |A(t)|=|M(t)−I|. Then the maximum root FD of the function h will implies FD is the multi-box fractal dimension FD of the multi- fuzzy fractal attractor of n objects for the just touching or non -overlapping set A∗∗. If|h(FD)| = |A(FD)|=|M(FD)−I| = 0 then FD is the multi-box dimension of the multi-fuzzy fractal attractor set A∗∗.

Therefore the matrix M(FD) has a non -trivial fixed point in Rn, while A(FD) has a non -trivial solution C of A(FD)C = 0 . Since there exists an FDsuch that the sum of the entries of each column of the matrix M(FD)−I is zero.This implies that the echelon matrix of A(FD) = M(FD)−I , has rank r < n, because; in this matrix when we use the row operation to add all rows, the nth row will be zero. So, the solution C can be obtained from the echelon matrix A, by the following approach.

$$\begin{pmatrix} 1 & x & x & 0 & x & \cdots & x & 0 & \cdots & 0 & x & \cdots & x \\ 0 & 0 & 0 & 1 & x & \cdots & x & 0 & \cdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & x & \cdots & x \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & x & \cdots & x \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\ 0 & \cdots & \cdots & & & \cdots & \cdots & & & 0 & 0 & \cdots & \cdots \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

After the rearrangement of this matrix we have

$$\begin{pmatrix} I & \cdots & B \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thereforeeach $C_i + \sum_{j=r+1}^{n} b_{ij} = 0$,whereC$_i$ = 1forall j = r+1,...,n, $C_i = -\sum_{j=r+1}^{n} b_{ij}$,whichimpliesthereexistsa non -trivial fixed point of the matrix M(FD) that further implies that |M(FD)−I| = 0. Then FD is a multi-box dimension of the multi-fuzzy fractal set.

For finding FD , let $f(t) = \sum_{i,j=1}^{n} \sum_{k} \left(r_{ij=1}^{*n}\right)^t$,is a decreasing function such that f (t) → 0, as t→ 0. Therefore thereexistsauniqueFD,suchthat $\sum_{i,j=1}^{n} \sum_{k} \left(r_{ij=1}^{*n}\right)^{FD} = 1$ whichisthemulti-boxfractaldimensionofthemulti-fuzzyfractal attractor sets

Conclusion

In this paper, we demonstrated there exists a new multi-box fractal dimension to the multi-fuzzy fractal attractor generated by IFS, by using matrix algebra .

Since there is no solution for the general polynomial of degree n≥5, we presented a theorem to find the multi- box dimension for the special condition of a multi-fuzzy fractal set generated by similarities containing n objects.

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