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ON THE POSITIVE PELL EQUATION $y^2 = 12x^2 + 16$

S.Devibala,

Department of Mathematics,

Sri Meenakshi Govt. Arts College for Women (A), Madurai, TamilNadu

ABSTRACT

The binary quadratic equation represented by the positive Pellian $y^2 = 12x^2 + 16$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabolas and special Pythagorean triangle.

Keywords: Binary quadratic, hyperbola, parabola, Integral solutions, Pell equation.
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INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is a non - square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 12x^2 + 16$ considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$y^2 = 12x^2 + 16 \qquad (1.1)$$

whose smallest positive integer solution is

$$x_0 = 2,\ y_0 = 8$$

To obtain the other solutions of (1.1), consider the Pell equation

$$y^2 = 12x^2 + 1$$

whose smallest positive integer solution is $(\widetilde{x_0}, \widetilde{y_0}) = (2,7)$

The general solution of (1.2) is given by

$$\widetilde{x_n} = \frac{1}{2\sqrt{12}} g_n, \widetilde{y_n} = \frac{1}{2} f_n,$$

where

$$f_n = (7 + 2\sqrt{12})^{n+1} + (7 - 2\sqrt{12})^{n+1}$$

$$g_n = (7+2\sqrt{12})^{n+1} - (7-2\sqrt{12})^{n+1}$$

Applying Brahmagupta lemma between $(\widetilde{x_0},\widetilde{y_0})$ and $(\widetilde{x_n},\widetilde{y_n})$, the other integer solutions of (1.1) are given by,

$$\sqrt{3}x_{n+1} = \sqrt{3}f_n + 2g_n$$

$$\sqrt{3}y_{n+1} = 4\sqrt{3}f_n + 6g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 14x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 14y_{n+2} + y_{n+3} = 0$$

Some numerical examples of x and y satisfying (1.1) are given in the following table below:

Table 1

n	x_{n+1}	y_{n+1}
0	2	8
1	30	104
2	418	1448
3	9854	36296
4	59378	215768

From the above table, we observe some interesting relations among the solutions which are presented below:

- Both x_n & y_n values are even
- **Each of the following expressions is a nasty number**

- $(3x_{2n+3} - 39x_{2n+2} + 12)$
- $\frac{1}{14}(3x_{2n+4} - 543x_{2n+2} + 168)$
- $(6y_{2n+2} - 18x_{2n+2} + 12)$
- $\frac{1}{17}(414x_{2n+2} - 6y_{2n+3} + 204)$
- $\frac{1}{239}(5778x_{2n+2} - 6y_{2n+4} + 2868)$
- $(39x_{2n+4} - 543x_{2n+3} + 12)$
- $\frac{1}{7}(78y_{2n+2} - 18x_{2n+3} + 84)$
- $\frac{1}{359}(414x_{2n+3} - 78y_{2n+3} + 4308)$
- $\frac{1}{5033}(5778x_{2n+3} - 78y_{2n+4} + 60396)$

- $\frac{1}{97}(1086y_{2n+2}-18x_{2n+4}+1164)$
- $\frac{1}{5009}(414x_{2n+4}-1086y_{2n+3}+60108)$
- $\frac{1}{70223}(5778x_{2n+4}-1086y_{2n+4}+842676)$
- $\frac{1}{280}(963y_{2n+2}-3y_{2n+4}+3360)$
- $\frac{1}{20}(69y_{2n+2}-3y_{2n+3}+240)$
- $\frac{1}{20}(963y_{2n+3}-69y_{2n+4}+240)$

➢ **Each of the following expressions is a cubical integer**

- $\frac{1}{2}[3x_{n+2}-39x_{n+1}+x_{3n+4}-13x_{3n+3}]$
- $\frac{1}{28}[3x_{n+3}-543x_{n+1}+x_{3n+5}-181x_{3n+3}]$
- $3y_{n+1}-9x_{n+1}+y_{3n+3}-3x_{3n+3}$
- $\frac{1}{17}[207x_{n+1}-3y_{n+2}+69x_{3n+3}-y_{3n+4}]$
- $\frac{1}{239}[2889x_{n+1}-3y_{n+3}+963x_{3n+3}-y_{3n+5}]$
- $\frac{1}{2}[39x_{n+3}-543x_{n+2}+13x_{3n+5}-181x_{3n+4}]$
- $\frac{1}{7}[39y_{n+1}-9x_{n+2}+13y_{3n+3}-3x_{3n+4}]$
- $\frac{1}{359}[207x_{n+2}-39y_{n+2}+69x_{3n+4}-13y_{3n+4}]$
- $\frac{1}{5033}[2889x_{n+2}-39y_{n+3}+963x_{3n+4}-13y_{3n+5}]$
- $\frac{1}{97}[543y_{n+1}-9x_{n+3}+181y_{3n+3}-3x_{3n+5}]$
- $\frac{1}{5099}[207x_{n+3}-543y_{n+2}+69x_{3n+5}-181y_{3n+4}]$
- $\frac{1}{70223}[2889x_{n+3}-543y_{n+3}+963x_{3n+5}-181y_{3n+5}]$
- $\frac{1}{560}[963y_{n+1}-3y_{n+3}+321y_{3n+3}-y_{3n+5}]$
- $\frac{1}{40}[69y_{n+1}-3y_{n+2}+23y_{3n+3}-y_{3n+4}]$
- $\frac{1}{40}[963y_{n+2}-69y_{n+3}+321y_{3n+4}-23y_{3n+5}]$

➢ **Each of the following expressions is a biquadratic number**

- $\frac{1}{2}[x_{4n+5}-13x_{4n+4}+4x_{2n+3}-52x_{2n+2}+12]$
- $\frac{1}{28}[x_{4n+6}-181x_{4n+4}+4x_{2n+4}-724x_{2n+2}+168]$
- $y_{4n+4}-3x_{4n+4}+4y_{2n+2}-12x_{2n+2}+6]$
- $\frac{1}{17}[69x_{4n+4}-y_{4n+5}+276x_{2n+2}-4y_{2n+3}+102]$
- $\frac{1}{239}[963x_{4n+4}-y_{4n+6}+3852x_{2n+2}-4y_{2n+4}+1434]$
- $\frac{1}{2}[13x_{4n+6}-181x_{4n+5}+52x_{2n+4}-724x_{2n+3}+12]$
- $\frac{1}{7}[13y_{4n+4}-3x_{4n+5}+52y_{2n+2}-12x_{2n+3}+42]$
- $\frac{1}{359}[69x_{4n+5}-13y_{4n+5}+276x_{2n+3}-52y_{2n+3}+2154]$
- $\frac{1}{5033}[963x_{4n+5}-13y_{4n+6}+3852x_{2n+3}-52y_{2n+4}+30198]$
- $\frac{1}{97}[181y_{4n+4}-3x_{4n+6}+724y_{2n+2}-12x_{2n+4}+582]$
- $\frac{1}{5009}[69x_{4n+6}-181y_{4n+5}+276x_{2n+4}-724y_{2n+3}+30054]$

- ❖ $\frac{1}{70223}[963x_{4n+6} - 181y_{4n+6} + 3852x_{2n+4} - 724y_{2n+4} + 421338]$
- ❖ $\frac{1}{560}[321y_{4n+4} - y_{4n+6} + 1284y_{2n+2} - 4y_{2n+4} + 3360]$
- ❖ $\frac{1}{40}[23y_{4n+4} - y_{4n+5} + 92y_{2n+2} - 4y_{2n+3} + 240]$
- ❖ $\frac{1}{40}[321y_{4n+5} - 23y_{4n+6} + 1284y_{2n+3} - 92y_{2n+4} + 240]$

Some relations satisfied by the solutions are follows

- ❖ $781x_{n+2} - 10879x_{n+1} - 4x_{n+3} = 0$
- ❖ $7x_{n+2} - 97x_{n+1} - 2y_{n+1} = 0$
- ❖ $121x_{n+2} - 1711x_{n+1} - 2y_{n+2} = 0$
- ❖ $1687x_{n+2} - 23557x_{n+1} - 2y_{n+3} = 0$
- ❖ $x_{n+3} + x_{n+1} - 14x_{n+2} = 0$
- ❖ $5009x_{n+1} - 17x_{n+3} - 28y_{n+2} = 0$
- ❖ $70223x_{n+1} - 239x_{n+3} - 28y_{n+3} = 0$
- ❖ $2y_{n+1} + 7x_{n+1} - x_{n+2} = 0$
- ❖ $28y_{n+1} - 97x_{n+1} - x_{n+3} = 0$
- ❖ $359x_{n+1} - 2y_{n+2} - 17x_{n+2} = 0$
- ❖ $120x_{n+1} - y_{n+2} - 17y_{n+1} = 0$
- ❖ $239y_{n+2} - 120x_{n+1} - 17y_{n+3} = 0$
- ❖ $5033x_{n+1} - 2y_{n+3} - 239x_{n+2} = 0$
- ❖ $1680x_{n+1} - y_{n+3} - 239y_{n+1} = 0$
- ❖ $7x_{n+3} - 97x_{n+2} - 2y_{n+1} = 0$
- ❖ $5009x_{n+2} - 359x_{n+3} - 2y_{n+2} = 0$
- ❖ $70223x_{n+2} - 5033x_{n+3} - 2y_{n+3} = 0$
- ❖ $120x_{n+2} - 359y_{n+1} - 7y_{n+2} = 0$
- ❖ $1680x_{n+2} - 5033y_{n+1} - 7y_{n+3} = 0$
- ❖ $5033y_{n+2} - 120x_{n+2} - 359x_{n+3} = 0$
- ❖ $120x_{n+3} - 5009y_{n+1} - 97y_{n+2} = 0$
- ❖ $1680x_{n+3} - 70223y_{n+1} - 97y_{n+3} = 0$
- ❖ $70223y_{n+2} - 120x_{n+3} - 5009y_{n+3} = 0$
- ❖ $y_{n+1} + y_{n+3} - 14y_{n+2} = 0$
- ❖ $28y_{n+1} - y_{n+2} - 60x_{n+1} = 0$
- ❖ $433y_{n+1} - 16y_{n+2} - 60x_{n+2} = 0$
- ❖ $6034y_{n+1} - 223y_{n+2} - 60x_{n+3} = 0$
- ❖ $12479y_{n+1} - 403y_{n+2} - 40y_{n+3} = 0$

3.Remarkable observations:

3.1:Employing linear combinations among the solutions of (1.1),one may generate integer solutions for other choices of hyperbola which are presented in the table 2 below:

Table 2

S.No	**Hyperbola**	$(\boldsymbol{Y}, \boldsymbol{X})$
1.	$Y^2 - 12X^2 = 192$	$(45x_{n+1} - 3x_{n+2}, x_{n+2} - 13x_{n+1})$

2.	$Y^2 - 12X^2 = 37632$	$(627x_{n+1} - 3x_{n+3}, x_{n+3} - 181x_{n+1})$
3	$Y^2 - 12X^2 = 48$	$(12x_{n+1} - 3y_{n+1}, y_{n+1} - 3x_{n+1})$
4.	$Y^2 - 12X^2 = 13872$	$(3y_{n+2} - 156x_{n+1}, 69x_{n+1} - y_{n+2})$
5.	$Y^2 - 12X^2 = 2741808$	$(3y_{n+3} - 2172x_{n+1}, 963x_{n1} - y_{n+3})$
6.	$Y^2 - 12X^2 = 192$	$(627x_{n+2} - 45x_{n+3}, 13x_{n+3} - 181x_{n+2})$
7.	$Y^2 - 12X^2 = 2352$	$(12x_{n+2} - 45y_{n+1}, 13y_{n+1} - 3x_{n+2})$
8.	$Y^2 - 12X^2 = 6186288$	$(45y_{n+2} - 156x_{n+2}, 69x_{n+2} - 13y_{n+2})$
9.	$Y^2 - 12X^2 = 1215892272$	$(45y_{n+3} - 2172x_{n+2}, 963x_{n+2} - 13y_{n+3})$
10.	$Y^2 - 12X^2 = 451632$	$(12x_{n+3} - 627y_{n+1}, 181y_{n+1} - 3x_{n+3})$
11.	$Y^2 - 12X^2 = 1204323888$	$(627y_{n+2} - 156x_{n+3}, 69x_{n+3} - 181y_{n+2})$
12.	$Y^2 - 12X^2 = 236700946992$	$(627y_{n+3} - 2172x_{n+3}, 963x_{n+3} - 181y_{n+3})$
13.	$Y^2 - 12X^2 = 15052800$	$(4y_{n+3} - 724y_{n+1}, 321y_{n+1} - y_{n+3})$
14.	$Y^2 - 12X^2 = 76800$	$(y_{n+2} - 13y_{n+1}, 23y_{n+1} - y_{n+2})$
15.	$Y^2 - 12X^2 = 76800$	$(52y_{n+3} - 724y_{n+2}, 321y_{n+2} - 23y_{n+3})$

3.2: Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of parabola which are presented in the table 3 below:

Table 3

S.No	Parabola	(Y,X)
1.	$Y^2 = 24X - 96$	$(x_{2n+3} - 13x_{2n+2}, 3x_{n+2} - 45x_{n+1})$
2.	$Y^2 = 336X - 18816$	$(x_{2n+4} - 181x_{2n+2}, 627x_{n+1} - 3x_{n+3})$
3.	$Y^2 = 12X - 24$	$(y_{2n+2} - 3x_{2n+2}, 12x_{n+1} - 3y_{n+1})$
4.	$Y^2 = 204X - 6936$	$(69x_{2n+2} - y_{2n+3}, 3y_{n+2} - 156x_{n+1})$
5.	$Y^2 = 2868X - 1370904$	$(963x_{2n+2} - y_{2n+4}, 3y_{n+3} - 2172x_{n+1})$

6.	$Y^2 = 24X - 96$	$(13x_{2n+4} - 181x_{2n+3}, 627x_{n+2} - 45x_{n+3})$
7.	$Y^2 = 84X - 1176$	$(13y_{2n+2} - 3x_{2n+3}, 12x_{n+2} - 45y_{n+1})$
8.	$Y^2 = 4308X - 3093144$	$(69x_{2n+3} - 13y_{2n+3}, 45y_{n+2} - 156x_{n+2})$
9.	$Y^2 = 60396X - 607946136$	$(963x_{2n+3} - 13y_{2n+4}, 45y_{n+3} - 2172x_{n+2})$
10.	$Y^2 = 1164X - 225816$	$(181y_{2n+2} - 3x_{2n+4}, 12x_{n+3} - 627y_{n+1})$
11.	$Y^2 = 60108X - 602161944$	$(69x_{2n+4} - 181y_{2n+3}, 627y_{n+2} - 156x_{n+3})$
12.	$Y^2 = 842676X - 118350473496$	$(963x_{2n+4} - 181y_{2n+4}, 627y_{n+3} - 2172x_{n+3})$
13.	$Y^2 = 6720X - 7526400$	$(321y_{2n+2} - y_{2n+4}, 4y_{n+3} - 724y_{n+1})$
14.	$Y^2 = 480X - 38400$	$(23y_{2n+2} - y_{2n+3}, y_{n+2} - 13y_{n+1})$
15.	$Y^2 = 480X - 38400$	$(321y_{2n+3} - 23y_{2n+4}, 52y_{n+3} - 724y_{n+2})$

3.3: Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}.$ Note that $p > q > 0$. Treat p, q as the generators of the Pythagorean triangle $T(X, Y, Z)$ where $X = 2pq, Y = p^2 - q^2, Z = p^2 + q^2$

Then the following results are obtained:

a) $X - 6Y + 5Z = -16$
b) $\frac{2A}{P} = x_{n+1}y_{n+1}$
c) $3(Z - Y)$ is a nasty number
d) $3(X - \frac{4A}{P})$ is a nasty number
e) $X - \frac{4A}{P} + Y$is written as the sum of two squares.

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