FURTHER RESULTS ON 3–REMAINDER CORDIAL LABELING OF GRAPHS

K. ANNATHURAI*
R. PONRAJ**
R. KALA***

ABSTRACT

Let G be a (p, q) graph. Let f be a function from V (G) to the set \{1, 2, \ldots, k\} where k is an integer \(2 < k \leq |V (G)|\). For each edge uv assign the label r where r is the remainder when f(u) is divided by f(v) (or) f(v) is divided by f(u) according as f(u) \(\geq f(v)\) or f(v) \(\geq f(u)\). Then the function f is called a k-remainder cordial labeling of G if \(|v_f (i) - v_f (j)| \leq 1, \ i, j \in \{1, \ldots, k\}\) \(v_f (x)\) denote the number of vertices labelled with x and \(|\eta_e (0) - \eta_o (1)| \leq 1\) respectively denote the number of edges labelled with an even integers and number of edges labelled with an odd integers. A graph admits a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the 3- remainder cordial labeling behavior of the subdivision of the star, wheel, subdivision of the wheel, subdivision of the comb, armed crown, fan, square of the path, \(K_{1,n} \odot K_2\), etc..

* Research Scholar, Department of Mathematics, Manonmaniam Sundaranar University, Abishekappati, Tirunelveli–627 012, Tamil Nadu, India.
** Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India, Affiliated By Manonmaniam Sundaranar University.
*** Department of Mathematics, Manonmaniam Sundaranar University, Abishekappati, Tirunelveli–627 012, Tamil Nadu, India.
1. INTRODUCTION

In this paper we considered only finite and simple graphs. Let $G_1$ and $G_2$ be two graphs with vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ respectively. Then their join $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv: u \in V_1$ and $v \in V_2\}$. The graph $W_n=C_n+K_1$ is called a wheel. In a wheel, a vertex of degree 3 is called a rim vertex. A vertex which is adjacent to all the rim vertices is called the central vertex. The edges with one end incident with the rim and the other incident with the central vertex are called spokes. The subdivision graph $S(G)$ of a graph $G$ is obtained by replacing each edge $uv$ by a path $uwv$. A comb is a caterpillar in which each vertex in the path is joined to exactly one pendant vertex. A graph $C_n \Theta K_1$ is called a crown. The corona of $G_1$ with $G_2$, $G_1 \odot G_2$ is the graph obtained by taking one copy of $G_1$ and $p_1$ copies of $G_2$ and joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$. Cahit [1], introduced the concept of cordial labeling of graphs. Ponraj et al. [4, 6], introduced remainder cordial labeling of graphs and investigate the remainder cordial labeling behavior of path, cycle, star, bistar, complete graph, etc., and also the concept of $k$-remainder cordial labeling introduced in [5]. Recently[9] they investigate the $3$-remainder cordial labeling behavior of some graphs. In this paper we investigate the $3$- remainder cordial labeling behavior of the subdivision of the star, wheel, subdivision of the wheel, subdivision of the comb, armed crown, fan, square of the path, $K_{1,n} \Theta K_2$, etc., Terms are not defined here follows from Harary [3] and Gallian [2].

2. $k$-REMAINDER CORDIAL LABELING

**Definition 2.1** : Let $G$ be a $(p, q)$ graph. Let $f$ be a function from $V (G)$ to the set $\{1, 2, \ldots , k\}$ where $k$ is an integer $2 < k \leq |V (G)|$. For each edge $uv$ assign the label $r$ where $r$ is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function $f$ is called a $k$-remainder cordial labeling of $G$ if $|v_f(i) − v_f(j)| \leq 1$, $i, j \in \{1, \ldots , k\}$ where $v_f(x)$ denote the number of vertices labeled with $x$ and $|\eta_e(0) − \eta_e(1)| \leq 1$ where $\eta_e(0)$ and $\eta_e(1)$ respectively denote the number of edges labeled with an even integers and number of edges labelled with an odd integers. A graph with a $k$- remainder cordial labeling is called a $k$-remainder cordial graph.

Now we investigate the $3$- remainder cordial labeling behavior of the wheel $W_n$.

**Theorem 2.2** : The wheel $W_n$ is 3-remainder cordial if and only if $n\equiv 1 (mod 3)$.

**Proof.** Let $W_n = C_n + K_1$, where $C_n$ is the cycle $u_1 u_2 \ldots , u_n u_1$ and $V(K_1) = \{u\}$. The proof of this theorem is proved in the following three cases.

Case(i): $n \equiv 0 (mod 3)$, $n \geq 3$.

Let $n = 3t$, $t > 1$. Suppose $f$ is a 3- remainder cordial labeling of the wheel.

Subcase(i):

Let $f(u) = 1$. Then all the spokes receives the label 0. Also to get the minimum possible zero in the rim edges, when 1’s are labeled consecutively. Therefore $\eta_e(0) \geq n + (t-1) + 2$
\[ n + t + 1 = 3t + t + 1 \]
\[ \eta_e(0) \geq 4t + 1, \text{ which is a contradiction to the size of } W_n. \]

**Subcase(ii):**
Let \( f(u) = 1. \) Then maximum possible 1’s appear in the rim edges when 2 and 3 should be labeled alternatively. That is 1’s consecutively.
\[ \eta_o(1) \leq 2t - 1 + t = 3t - 1 \]
\[ \eta_o(1) \leq 3t - 1, \text{ which is a contradiction to the size of } W_n. \]

**Subcase(iii):**
Let \( f(u) = 3. \) Then similar to subcase(ii), we get a contradiction.

**Case (i):** \( n \equiv 1 \pmod{3} \)
Fix the label 2 to the central vertex \( u \) of the wheel. Next assign the labels 3 to the vertices \( u_1, u_3, \ldots, u_{(2n+1)/3} \). Then assign the label 2 to the vertices \( u_2, u_4, \ldots, u_{(2n-2)/3} \). Finally assign the label 1 to the vertices \( u_{2n+4}/3, u_{(2n+4)/3+1}, \ldots, u_n \). The **Table 1** establish that this vertex function \( f \) is 3-remainder cordial labeling of \( W_n \) for all \( n \).

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( \nu_f(1) )</th>
<th>( \nu_f(2) )</th>
<th>( \nu_f(3) )</th>
<th>( \eta_e(0) )</th>
<th>( \eta_o(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 1 \pmod{3} )</td>
<td>( n - 1 ) ( 3 )</td>
<td>( n + 2 ) ( 3 )</td>
<td>( n + 2 ) ( 3 )</td>
<td>( n )</td>
<td>( n )</td>
</tr>
</tbody>
</table>

**Table 1**

**Case (iii):** \( n \equiv 2 \pmod{3} \)
Proceed as in case(i), we get a contradiction.

Next we investigate the subdivision of the wheel \( W_n \).

**Theorem 2.3:** The graph \( S(W_n) \) is 3-remainder cordial for all values of \( n \).

**Proof.** Let \( V(S(W_n)) = \{u, u_i, v_i, w_i : 1 \leq i \leq n\} \) and \( E(S(W_n)) = \{uu_i, u_iv_i, v_iw_i : 1 \leq i \leq n\} \) U \( \{w_iV_{i+1} : 1 \leq i \leq n-1\} \) U \( \{w_nv_n\} \). Then the subdivision of the wheel has \( 3n+1 \) vertices and \( 4n \) edges.
Fix the label 2 to the central vertex \( u \) of wheel. Next assign the labels to the vertices \( u_i, v_i, w_i \) for \( 1 \leq i \leq n \) as follows.
- \( f(u_i) = 3 \) for \( i = 1 \) to \( n \),
- \( f(v_i) = 2 \) for \( i = 1 \) to \( n \),
- \( f(w_i) = 1 \) for \( i = 1 \) to \( n \).

Note that \( \nu_f(1)=\nu_f(3)=n, \nu_f(2)=n+1 \) and \( \eta_f(0)=\eta_f(1)=2n \). Then clearly \( S(W_n) \) is 3-remainder cordial for all values of \( n \).

Now we investigate the subdivision of the comb.

**Theorem 2.4:** The graph \( S(P_n \circ K_1) \) is 3-remainder cordial for all \( n \).
**Proof.** Let $P_n$ be a path $u_1, u_2, \ldots, u_n$. Let $V(S(P_n \circ K_1)) = \{ u_i, v_i, w_i : 1 \leq i \leq n \} \cup \{ u_i' : 1 \leq i \leq n-1 \}$ and $E(S(P_n \circ K_1)) = \{ u_iv_i, v_iw_i : 1 \leq i \leq n \} \cup \{ u_iu_i', u_i'w_{i+1} : 1 \leq i \leq n-1 \}$. Then the subdivision of the comb has $4n-1$ vertices and $4n-2$ edges.

**Case(i):** $n \equiv 0 \pmod{3}$

Assign the labels to the vertices $u_i$ for $1 \leq i \leq n$ as follows.

\[
f(u_i) = \begin{cases} 
1, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\
3, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 1. \\
2, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n - 3.
\end{cases}
\]

and assign the label 1 to the vertex $u_n$.

Next assign the labels to the vertices $v_i$, for $1 \leq i \leq n$ as follows.

\[
f(v_i) = \begin{cases} 
1, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\
2, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 1. \\
3, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n.
\end{cases}
\]

and assign the labels to the vertices $w_i$, for $1 \leq i \leq n$ as follows.

\[
f(w_i) = \begin{cases} 
1, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\
3, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 1. \\
2, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n.
\end{cases}
\]

and then assign the labels to the vertices $u_i'$, for $1 \leq i \leq n$ as follows.

\[
f(u_i') = \begin{cases} 
2, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\
3, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 4. \\
1, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n - 3.
\end{cases}
\]

Finally assign the label 2 to the vertices $u_{n-1}'$.

**Case(ii):** $n \equiv 1 \pmod{3}$

Assign the labels to the vertices $u_1, u_2, \ldots, u_{n-1}$ as follows.

\[
f(u_i) = \begin{cases} 
3, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 3. \\
2, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 2. \\
1, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n - 1.
\end{cases}
\]

and assign the label 2 to the vertex $u_n$.

Next assign the labels to the vertices $u_1', u_2', \ldots, u_{n-1}'$ as follows.

\[
f(u_i') = \begin{cases} 
2, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\
3, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 4. \\
1, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n - 3.
\end{cases}
\]

As in case(i) assign the labels to the vertices $v_i$, $w_i$ for $1 \leq i \leq n-1$.

Finally assign the labels 1 and 3 to the vertices $v_n$ and $w_n$ of the subdivision of the comb respectively.

**Case(iii):** $n \equiv 2 \pmod{3}$

Assign the labels to the vertices $u_1, u_2, \ldots, u_{n-2}$ as follows.
\[
\begin{align*}
\{1, & \text{ for } i = 1, 4, \ldots, i + 3, \ldots, n - 3. \\
f(u_i) = & \begin{cases} \\
3, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 2. \\
2, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n - 1. \\
\end{cases}
\end{align*}
\]
and assign the labels 1 and 3 to the vertices \(u_{n-1}\) and \(u_n\) respectively.

Next assign the labels to the vertices \(u_1', u_2', \ldots, u_{n-2}'\) as follows.
\[
f(u_i') = \begin{cases} \\
2, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\
1, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 4. \\
3, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n - 3. \\
\end{cases}
\]
and assign the label 2 to the vertices \(u_{n-1}'\).

As in case(i) assign the labels to the vertices \(v_1, w_i\) for \(1 \leq i \leq n-2\).
Finally assign the labels 1, 2; 1, 3 respectively to the vertices \(v_{n-1}, v_n; w_{n-1}, w_n\) of the subdivision of the comb respectively. Thus the table 2, given below establish that this vertex labeling \(f\) is 3-remainder cordial labeling of \(S(P_n \circ K_1)\) for all \(n\).

<table>
<thead>
<tr>
<th>Nature of (n)</th>
<th>(v_1(1))</th>
<th>(v_1(2))</th>
<th>(v_1(3))</th>
<th>(\eta_e(0))</th>
<th>(\eta_o(1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n \equiv 0(\mod 3))</td>
<td>(\frac{4n}{3})</td>
<td>(\frac{4n}{3})</td>
<td>(4n-3)</td>
<td>2n-1</td>
<td>2n-1</td>
</tr>
<tr>
<td>(n \equiv 1(\mod 3))</td>
<td>(\frac{4n-1}{3})</td>
<td>(\frac{4n-1}{3})</td>
<td>(4n-1)</td>
<td>2n-1</td>
<td>2n-1</td>
</tr>
<tr>
<td>(n \equiv 2(\mod 3))</td>
<td>(\frac{4n+1}{3})</td>
<td>(\frac{4n-2}{3})</td>
<td>(4n-2)</td>
<td>2n-1</td>
<td>2n-1</td>
</tr>
</tbody>
</table>

Next we investigate the subdivision of the star \(S(K_{1,n})\).

**Theorem 2.5**: The graph \(S(K_{1,n})\) is 3-remainder cordial for all \(n\).

**Proof**. Let \(u\) be \(n^{th}\) degree vertex of \(S(K_{1,n})\). Let \(V(S(K_{1,n})) = \{u, u_i, v_i : 1 \leq i \leq n\}\) and \(E(S(K_{1,n})) = \{ uu_i, u_i v_i : 1 \leq i \leq n\}\). Note that the subdivision of the star has \(2n+1\) vertices and \(2n\) edges.

Fix the label 2 to the central vertex \(u\) in the following all cases.

**Case(i)**: \(2n\equiv 0(\mod 3)\)
Assign the label 3 to the vertices \(u_1, u_2, u_3, \ldots, u_{2n/3}\) consecutively. Then assign the label 2 to the remaining vertices \(u_{(2n/3)+1}, u_{(2n/3)+2}, \ldots, u_n\). Next assign the label 2 to the pendant vertices \(v_1, v_2, v_3, \ldots, v_{2n/6}\) consecutively and assign the label 1 to the remaining pendant vertices \(v_{(2n/6)+1}, v_{(2n/6)+2}, \ldots, v_n\).

**Case(ii)**: \(2n\equiv 1(\mod 3)\)
Assign the label 3 to the vertices \(u_1, u_2, u_3, \ldots, u_{(2n+2)/3}\) consecutively. Then next assign the label 2 to the remaining vertices \(u_{((2n+2)/3)+1}, u_{((2n+2)/3)+2}, \ldots, u_n\). Now assign the label 2 to the pendant vertices \(v_1, v_2, v_3, \ldots, v_{(n-2)/3}\) consecutively and 1 to the remaining pendant vertices \(v_{((n-2)/3)+1}, v_{((n-2)/3)+2}, \ldots, v_n\).

**Case(iii)**: \(2n\equiv 2(\mod 3)\)
Assign the label 3 to the vertices \(u_1, u_2, u_3, \ldots, u_{(2n+1)/3}\). Then next assign the label 2 to the vertices \(u_{(2n+1)/3}+1, u_{(2n+1)/3}+2, \ldots, u_n\). Next assign the label 2 to the pendant vertices \(v_1, v_2, v_3, \ldots, v_{(n-1)/3}\) consecutively and 1 to the remaining pendant vertices \(v_{(n-1)/3}+1, v_{(n-1)/3}+2, \ldots, v_n\). The table-3 shows that this vertex function \(f\) is 3- remainder cordial labeling of the subdivision of the star \(S(K_{1,n})\) for all \(n\).

<table>
<thead>
<tr>
<th>Nature of (n)</th>
<th>(v_1(1))</th>
<th>(v_1(2))</th>
<th>(v_1(3))</th>
<th>(\eta_c(0))</th>
<th>(\eta_o(1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2n+1 \equiv 0 \pmod{3})</td>
<td>(\frac{n}{3})</td>
<td>(\frac{n+1}{3})</td>
<td>(\frac{n+2}{3})</td>
<td>(n)</td>
<td>(n-1)</td>
</tr>
<tr>
<td>(2n+1 \equiv 1 \pmod{3})</td>
<td>(\frac{n-1}{3})</td>
<td>(\frac{n+2}{3})</td>
<td>(\frac{n+3}{3})</td>
<td>(n-1)</td>
<td>(n)</td>
</tr>
<tr>
<td>(2n+1 \equiv 2 \pmod{3})</td>
<td>(\frac{n+1}{3})</td>
<td>(\frac{n+1}{3})</td>
<td>(\frac{n+2}{3})</td>
<td>(n)</td>
<td>(n-1)</td>
</tr>
</tbody>
</table>

Table-3

Here we investigate the fan \(F_n\).

**Theorem 2.5:** The fan is 3- remainder cordial for all values \(n\).

**Proof.** Let \(F_n = P_n + K_1\) where \(P_n\) is a path \(u_1, u_2, \ldots, u_n\) of length \(n\) and \(V(K_1)=\{u\}\). Let \(V(P_n+K_1) = V(P_n) \cup \{u\}\) and \(E(P_n+K_1) = \{uu_i : 1 \leq i \leq n\} \cup \{u_{(i+1)} : 1 \leq i \leq n-1\}\). Let \(V(S(K_{1,n})) = \{u, v_i : 1 \leq i \leq n\}\) and \(E(S(K_{1,n})) = \{uu_i, u_i v_i : 1 \leq i \leq n\}\). Clearly the graph \(P_n+K_1\) has \(n+1\) vertices and \(2n-1\) edges.

Fix the label 2 to the central vertex \(u\) of \(K_1\) in the following three cases.

**Case(i):** \(n \equiv 0 \pmod{3}\)

Assign the label 1 to the vertices \(u_1, u_2, u_3, \ldots, u_{n/3}\) and assign the label 3 to the vertices \(u_{(n/3)+1}, u_{(n/3)+2}, \ldots, u_n\). Next then assign the label 2 to the vertices \(u_{(n/3)+2}, u_{(n/3)+4}, \ldots, u_{n-1}\).

**Case(ii):** \(n \equiv 1 \pmod{3}\)

Assign the label 1 to the vertices \(u_1, u_2, u_3, \ldots, u_{(n-1)/3}\) and 3 to the vertices \(u_{((n-1)/3)+1}, u_{((n-1)/3)+2}, \ldots, u_n\) and then assign the label 2 to the vertices \(u_{((n-1)/3)+2}, u_{((n-1)/3)+4}, \ldots, u_{n-1}\).

**Case(iii):** \(n \equiv 2 \pmod{3}\)

Assign the label 1 to the vertices \(u_1, u_2, u_3, \ldots, u_{(n+1)/3}\) and assign the label 3 to the vertices \(u_{((n+1)/3)+1}, u_{((n+1)/3)+2}, \ldots, u_n\). Then assign the label 2 to the vertices \(u_{((n+1)/3)+2}, u_{((n+1)/3)+4}, \ldots, u_{n-1}\). The table-4 establish that this vertex function \(f\) is 3- remainder cordial labeling of the fan for all \(n\).

<table>
<thead>
<tr>
<th>Nature of (n)</th>
<th>(v_1(1))</th>
<th>(v_1(2))</th>
<th>(v_1(3))</th>
<th>(\eta_c(0))</th>
<th>(\eta_o(1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2n+1 \equiv 0 \pmod{3})</td>
<td>(\frac{n}{3})</td>
<td>(\frac{n+1}{3})</td>
<td>(\frac{n+2}{3})</td>
<td>(n)</td>
<td>(n-1)</td>
</tr>
<tr>
<td>(2n+1 \equiv 1 \pmod{3})</td>
<td>(\frac{n-1}{3})</td>
<td>(\frac{n+2}{3})</td>
<td>(\frac{n+3}{3})</td>
<td>(n-1)</td>
<td>(n)</td>
</tr>
<tr>
<td>(2n+1 \equiv 2 \pmod{3})</td>
<td>(\frac{n+1}{3})</td>
<td>(\frac{n+1}{3})</td>
<td>(\frac{n+2}{3})</td>
<td>(n)</td>
<td>(n-1)</td>
</tr>
</tbody>
</table>
Next we investigate the square of the path $P_n^2$.

**Theorem 2.6:** The square of the path $P_n^2$ is 3- remainder cordial for all values $n$.

**Proof.** Let $P_n$ be a path $u_1, u_2, \ldots, u_n$ of length $n$. Let $V(P_n^2) = \{u_i : 1 \leq i \leq n\}$ and $E(P_n^2) = \{u_iu_{i+2} : 1 \leq i \leq n-2\}$. Note that the square of the path has $n$ vertices and $2n-3$ edges.

**Case(i):** $n \equiv 0 \pmod{3}$

Assign the label 1 to the first two vertices $u_1$ and $u_2$, respectively to the last two vertices $u_{n-2}$ and $u_n$ of length $n$. Next assign the labels 2 and 3 to the first two vertices $u_{(n/3)+1}$ and $u_{(n/3)+2}$ respectively. Next assign the labels 3 and 2 respectively to the next two vertices $u_{(n/3)+3}$ and $u_{(n/3)+4}$. Then assign the labels 2 and 3 to the vertices $u_{(n/3)+5}$ and $u_{(n/3)+6}$ and so on. When $n$ is even, assign the labels 3 and 2 respectively to the last two vertices $u_{n-1}$ and $u_n$ respectively. When $n$ is odd, assign the labels 2 and 3 respectively to the last two vertices $u_{n-1}$ and $u_n$ of the square of the path respectively.

**Case(ii):** $n \equiv 1 \pmod{3}$

Assign the label 1 to the consecutive vertices $u_1, u_2, u_3, \ldots, u_{(n-1)/3}$. Next assign the labels 2 and 3 to the first two vertices $u_{(n-1)/3+1}$ and $u_{(n-1)/3+2}$ respectively. Then assign the labels 3 and 2 respectively to the next two vertices $u_{(n-1)/3+3}$ and $u_{(n-1)/3+4}$. Then next assign the labels 2 and 3 to the vertices $u_{(n-1)/3+5}$ and $u_{(n-1)/3+6}$ and so on. Proceeding like this until we reach the vertex $u_{n-1}$. Clearly the vertex $u_{n-1}$ is received the label 3. Finally assign the label 3 to the last vertex $u_n$ when $n$ is even. Otherwise assign the label 2 to the last vertex $u_n$ of the path.

**Case(iii):** $n \equiv 2 \pmod{3}$

Assign the label 1 to the first $(\frac{n-2}{3})$ vertices $u_1, u_2, u_3, \ldots, u_{(n-2)/3}$. Next assign the labels 2 and 3 to the first two vertices $u_{(n-2)/3+1}$ and $u_{(n-2)/3+2}$ respectively. Then assign the labels 3 and 2 respectively to the next two vertices $u_{(n-2)/3+3}$ and $u_{(n-2)/3+4}$. Then next assign the labels 2 and 3 to the next two vertices $u_{(n-2)/3+5}$ and $u_{(n-2)/3+6}$ and so on. Finally assign the labels 2 and 3 to the last two vertices $u_{n-1}$ and $u_n$ when $n$ is even. Otherwise assign the labels 3 and 2 to the last two vertices $u_{n-1}$ and $u_n$ of the path. The table-5 given below that this vertex function $f$ is 3- remainder cordial labeling of $P_n^2$ for all $n$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$\eta_e(0)$</th>
<th>$\eta_o(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{3}$</td>
<td>$n/3$</td>
<td>$n/3$</td>
<td>$n/3$</td>
<td>$n-1$</td>
<td>$n-2$</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{3}$ (i). $n$ is even</td>
<td>$n-1/3$</td>
<td>$n-1/3$</td>
<td>$n+2/3$</td>
<td>$n-1$</td>
<td>$n-2$</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{3}$ (ii). $n$ is odd</td>
<td>$n-1/3$</td>
<td>$n+1/3$</td>
<td>$n+1/3$</td>
<td>$n-2$</td>
<td>$n-1$</td>
</tr>
</tbody>
</table>

**Table –5**

Here we investigate the armed crown $AC_n$.
Theorem 2.7: The graph $AC_n$ is 3-remainder cordial for all values of $n$.

Proof. Let $C_n$ be a cycle $u_1u_2, \ldots, u_n u_1$ of length $n$. Let $V(AC_n) = V(C_n) \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(AC_n) = \{u_iv_i, v_iw_i : 1 \leq i \leq n\}$. It is easy to verify that the armed crown has $3n$ vertices and $3n$ edges.

Case(i): $n \equiv 0 \pmod{3}$
First assign the labels to the vertices $u_i, v_i$ for all $i=1$ to $n$ as follows.

$$f(u_i) = \begin{cases} 1, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\ 2, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 1. \\ 3, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n. \end{cases}$$

and $f(v_i) = \begin{cases} 1, & \text{for } i = 1, 4, \ldots, i + 3, \ldots, n - 2. \\ 2, & \text{for } i = 2, 5, \ldots, i + 3, \ldots, n - 1. \\ 3, & \text{for } i = 3, 6, \ldots, i + 3, \ldots, n. \end{cases}$

Next we consider the vertices $w_i$, $1 \leq i \leq n$ as follows.

Subcase(i): $n$ is odd
Assign the labels 1, 2, and 3 to the vertices $w_1$, $w_2$, and $w_3$ respectively. Next assign the labels 2, 1 and 3 respectively to the vertices $w_4$, $w_5$ and $w_6$. Then assign the labels 1, 2, and 3 to the vertices $w_7$, $w_8$ and $w_9$ respectively. Then next assign the labels 2, 1 and 3 respectively to the vertices $w_{10}$, $w_{11}$ and $w_{12}$. Continuing like this until we reach the vertices $w_{n-2}$, $w_{n-1}$ and $w_n$. Clearly in this process the vertices $w_{n-2}$, $w_{n-1}$ and $w_n$ are received the labels 1, 2 and 3 respectively.

Subcase(ii): $n$ is even.
As in subcase(i) assign the labels to the vertices $w_i$, $(1 \leq i \leq n-3)$. Finally assign the labels 2, 1 and 3 respectively to the vertices $w_{n-2}$, $w_{n-1}$ and $w_n$.

Case(ii): $n \equiv 1 \pmod{3}$
Assign the labels to the vertices $u_i$ and $v_i$, $(1 \leq i \leq n-1)$ as in case(i). Next assign the labels 2 and 1 to the vertices $u_6$ and $v_n$ respectively. Next assign the labels 1, 2, and 3 to the vertices $w_1$, $w_2$, and $w_3$ respectively. Then assign the labels 1, 2 and 3 respectively to the vertices $w_4$, $w_5$ and $w_6$ and so on. Proceeding like this until we reach the vertex $w_{n-1}$. Note that in this process the vertices $w_{n-3}$, $w_{n-2}$ and $w_{n-1}$ are received the labels 1, 2 and 3 respectively. Finally assign the label 3 to the vertex $w_n$ of the armed crown $AC_n$.

Case(iii): $n \equiv 2 \pmod{3}$
Assign the labels to the vertices $u_i$ and $v_i$, $(1 \leq i \leq n-2)$ as in case(i). Next assign the labels 2,1 and 3,1 to the vertices $u_{n-1}$, $u_n$ and $v_{n-1}$, $v_n$ respectively. Now we consider the vertices $w_i$, $(1 \leq i \leq n)$. Assign the labels 2, 1 and 3 to the vertices $w_1$, $w_2$, and $w_3$ respectively. Then assign the labels 1, 2 and 3 respectively to the vertices $w_4$, $w_5$ and $w_6$. Next assign the labels 2, 1 and 3 to the vertices $w_7$, $w_8$, and $w_9$ respectively and 1, 2 and 3 respectively to the vertices $w_{10}$, $w_{11}$ and $w_{12}$. In this way we reach the vertex $w_{n-5}$. Note that in this process the vertices $w_{n-7}$, $w_{n-6}$ and $w_{n-5}$ are received the labels 2, 1 and 3 respectively if $n$ is even. Otherwise the vertices $w_{n-7}$, $w_{n-6}$ and $w_{n-5}$ are received the labels 2, 1 and 3 respectively. Finally assign the labels 1, 2, 3, 2 and 3 to
the vertices $w_{n-4}$, $w_{n-3}$, $w_{n-2}$, $w_{n-1}$ and $w_n$ of the armed crown $AC_n$. Thus the table 6, given below establish that this vertex labeling $f$ is 3- remainder cordial labeling of the armed crown $AC_n$ for all $n$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
<th>$\eta_e(0)$</th>
<th>$\eta_o(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0,1,2 \pmod{3}$ (i). $n$ is odd</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$3n - 1$</td>
<td>$3n + 1$</td>
</tr>
<tr>
<td>$n \equiv 0,1,2 \pmod{3}$ (i). $n$ is even</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$3n/2$</td>
<td>$3n/2$</td>
</tr>
</tbody>
</table>

Table – 6

Finally we investigate the 3- remainder cordial labeling behavior of $K_{1,n} \circ K_2$.

**Theorem 2.8:** The graph $K_{1,n} \circ K_2$ is 3- remainder cordial for all $n$.

**Proof.** Let $V(K_{1,n} \circ K_2) = \{x, y, u, v, w_i : 1 \leq i \leq n\}$ and $E(K_{1,n} \circ K_2) = \{xy, xu, yu, uu_i, u_iv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\}$. Then the graph $K_{1,n} \circ K_2$ has $4n-1$ vertices and $4n-2$ edges.

Fix the labels 3,2 and 3 to the vertices $u$, $x$ and $y$ respectively and 2 to the vertices $u_1$, $u_2$, $u_3$, …, $u_n$ into the following two cases.

**Case(i):** $n$ is odd.

Now assign the labels to the vertices $v_i$, $w_i$ for $(1 \leq i \leq n)$ as follows.

$$f(v_i) = \begin{cases} 1, & \text{if } i = 1 \text{ to } (n+1)/2 \\ 3, & \text{if } i = ((n+1)/2) + 1 \text{ to } n \end{cases}$$

and $f(w_i) = \begin{cases} 1, & \text{if } i = 1 \text{ to } (n+1)/2 \\ 3, & \text{if } i = ((n+1)/2) + 1 \text{ to } n \end{cases}$

**Case(ii):** $n$ is even.

Assign the labels to the vertices $v_i$, $w_i$ ; $(1 \leq i \leq n)$ as follows.

$$f(v_i) = \begin{cases} 1, & \text{if } i = 1 \text{ to } (n+2)/2 \\ 3, & \text{if } i = ((n+2)/2) + 1 \text{ to } n \end{cases}$$

and $f(w_i) = \begin{cases} 1, & \text{if } i = 1 \text{ to } n/2 \\ 3, & \text{if } i = (n/2) + 1 \text{ to } n \end{cases}$

Note that the vertex condition and edge condition are $v_f(1)$= $v_f(2)$=$v_f(3)$=$n+1$ and $\eta_e(0)$=$2n+2$, $\eta_o(1)$=$2n+1$ respectively in the both cases of $n$. Hence the function $f$ is 3- remainder cordial labeling behavior of the graph $K_{1,n} \circ K_2$ for all $n$.

For illustration, a 3- remainder cordial labeling of $K_{1,5} \circ K_2$ is shown in Figure 2.1.
REFERENCES


