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DIFFERENT TYPES OF TOPOLOGIES IN FUNCTION SPACES

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Introduction:-

A function space is a topological space whose points are functions.There are many different kinds of function spaces and there are usually several different topologies that can be placed on a given set of functions.We describe three topologies.That can be placed on the set of all function from a set X to a space Y.

The three Topologies are:-

1. The Product topology
2. The Box topology
3. The Uniform topology

We also describe and study that a function space topology helps us to prove various useful approximation. Here the weak and strong topologies are introduced using co-ordinate based definition and sum of their basic properties.

We give some of the basic major approximation results.We introduce jets to give equivalent definitions of the weak and strong topology. As an application this definition is useful to prove another approximation results.

Set of Function:-

If X and Y are sets let Y^xdenote the set of all function from X to Y.

Example [1]: Let Y be a set and Let X = { x_1, ---------- ----- -------, x_n} be a finite set with n elements. The set Y^x consist of all function { x_1, ---------- ----- -------, x_n}→Y. Any such function can be thought of as an n tuple of points in Y:

$$f = (f(x_1), f(x_2), \dots \dots \dots \dots \dots \dots., \quad f(x_n))$$

Thus we can identify Y^x with the Cartesian power Y^n = Y× ------------ ×Y.

In fact, the n^{th} Cartesian power Y^n is sometimes defined as the set $Y^{(1\dots\dots\dots n)}$ of all functions {1, ………….., n} →Y. Using this definition, every ordered n-tuple (y_1, ……………………., y_n) is actually a function, withyk being an alternative notation for y(k).

Point wise Convergence:-

We are used to the idea of a sequence $x_n$of real number converging to some real numbers x. Generally, we know that it means for a sequence x_n of points in a topological space to converge to a point x.

Definition: Point wise Convergence

Let X be a set and Y be a topological space and Let f_n:X® Y be a sequence of functions then f_n converge point wise to a functionf$_n$:X® Y if for every $xI\ X$ the

sequencef$_n$(x) in Y. The sequence of functions f_n converges point wise to f if $\lim_{x®¥} fn(x) = f(x)$
f all values of X

Example [2]: The functions

$$fn(x) = n\sin\frac{x}{n}$$

From point wise convergence function $f(x) = x$

In particular

$$\lim_{n\to\infty} n\sin\frac{x}{n} = x$$

For every $x \in [0,2\pi]$

Example [3]: Here is an example involving functions¥ → i, which we write as infinite tuples, consider the following sequence of function,

$$f_1 = (1,2,3,4,5, \dots \dots \dots ..)$$
$$f_2 = (\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \dots \dots \dots ..)$$
$$f_3 = (\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \dots \dots \dots ..)$$

For any fixed kI¥, the sequence $f_n(k)$ consist of number $\frac{k}{n}$, and thus convers to 0. Therefore the function f_n converge point wise to the zero function;

$$f = (0,0,0,0,0, \dots \dots \dots \dots \dots \dots \dots \dots .)$$

The product Topology are define a topology on Y^x under which convergence of sequence corresponds to point wise convergence of functions.

<u>Definition: The Product Topology</u>

Let X be a set, and Y be a topological space. Given any $x \in X$ and any open set $U \subset Y$, define $s(x, U) = \{f \in Y^X / f(x) \in U\}$

Then the set $s(x, U)$ from a sub basis for a topology onY^X, known as the product topology.

The following example illustrate product topology agrees with the product topology for the cartesion product of two sets.

Example:-If Y is topological space then the product Y x Y can be viewed as a function space Y, where X = {1,2}. If UI Y is open, then

S(1,U) = U x Y

S(2,U) = Y x U

This form are sub basis for the product topology on Y X Y. The definition above agrees with our existing definition of the product topology.

Beware that the sets S(x,U) aresub basis for the product topology not a basis. A basic open set would be a finite intersection of some basic open sets: $S(x_1, U_1) \cap \ldots \ldots \ldots \ldots \ldots \cap S(x_n, U_n)$

Because this intersection is finite a basic open set can include restriction on only finitely many different function values. All the power definition of a sub basicS(x,u) involves an arbitrary open set U, you it is often helpful to restrict to the case where you is a basic open set.

Theorem:Sub basis for the product topology

Let X be a set, Y be a topological space, and let β be a basis for the topology on Y then the collection $\{S(x, B) | x \in X, S, B \in \beta\}$ is a sub basis for the product topology on Y^x.

Proof:-

Consider an element S (x,U) of the standard sub basis for the product topology. Then U is an open subset of Y, so U can be expressed as the union of some family $\{B_\propto\}_\propto \in$ J of elements of β.

Therefore,

$$S(x, U) = \bigcup_{a \hat{I} J} S(x, B_a)$$

Which proves S(x,U) lies in the topology generated by the sets S(x,B).

Hence the theorem

Theorem:Convergence in the Product Topology

Proof:- At first suppose that,

f_n Converges to f under the product topology, and let $x \in X$. If U is a neighbourhood of $f(x)$ in Y, then S(x,u) is a neighbourhood of f in Y^X.

So, $f_n \in S(x, U)$ for all but finitely many n. It follows that $f_n(x) \in U$ for all but finitely many n, which proves that

$$f_n(x) \to f(x)$$

For the converse, suppose that f_n converges point wise to f, and let $S(x, U)$ be a

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neighbourhood of f in Y^X. Then U is a neighbourhood of $f(x)$ in Y. Since $f_n(x) \to f(x)$ it follows that $f_n(x) \in U$ for all but finitely many n. Then $f_n \in S(x, U)$ for all but finitely many n, which proves that $f_n \to f$ under the product topology. Thus we have proved the theorem.

The Box Topology

Now we discuss about an important of topology known as the box topology.

Definition: The Box Topology

Let X be a set and Y be a topological space. Given a family $\{U_x\}_{x\in X}$ of open set in Y, the product $\widetilde{\text{O}}_{x\hat{I}X} U_x \ = \{f\hat{I}Y^X \big| f(x)\hat{I}U_x \, for \ every \ x\hat{I}X\}$is called an open box in Y^X.The collection of all open boxes from a basis for a topology on Y^X, known as the box topology.

Example:-

Consider again the function space $i^{¥}$ for any sequence of open intervals (a_1, b_1), (a_2, b_2), ………………… in i, the set (a_1, b_1) x (a_2, b_2) x (a_3, b_3), ……….. is an example of a basic open set in the box topology.Note that such a set is not open in the product topology.

Though the box topology mayseen more natural than the product topology it not actually very useful. In particular, very few sequence of function converse in the box topology.

Theorem:A Non –Metrizable space

Proof:-

We give an argument involving neighbourhood of the origin. Munkres for a proof involving sequence.

Suppose that d in a metric on R^N whose corresponding metric topology where the same as the box topology. Let O the donate the zero function (0, 0, 0, ...) in R^N, and consider the following sequence of open balls.

$$B_d(0,1) \supset \ B_d\left(0, \frac{1}{2}\right) \supset B_d\left(0, \frac{1}{3}\right) \dots \dots \dots \dots$$

By assumption, each of these balls is open in the box topology.So each balls$B_d\left(0, \frac{1}{n}\right)$ must contain a basic open box around 0. Thus, there exists a positive real number a_{ij} such that:

$$B_d(0,1) \supset (-a_{11}, a_{11}) \times \ (-a_{12}, a_{12}) \times \ (-a_{13}, a_{13}) \times \dots \dots \dots \dots \dots \dots$$

$$B_d\left(0, \frac{1}{2}\right) \supset (-a_{21}, a_{21}) \times \ (-a_{22}, a_{22}) \times \ (-a_{23}, a_{23}) \times \dots \dots \dots \dots \dots \dots$$

$$B_d\left(0, \frac{1}{3}\right) \supset (-a_{31}, a_{31}) \times \ (-a_{32}, a_{32}) \times \ (-a_{33}, a_{33}) \times \dots \dots \dots \dots \dots \dots$$

Without loss of generality, we may assume that the intervals in each column are

shrinking, i. e. That $a_{1k} \geq a_{2k} \geq a_{3k} \geq \cdots \ldots \ldots \ldots \ldots \ldots ..$ for each k.
Now consider the open box formed by the intervals along the diagonal

$$(-a_{11}, a_{11}) \times (-a_{22}, a_{22}) \times (-a_{33}, a_{33}) \times \ldots \ldots \ldots \ldots \ldots \ldots$$

This set is a neighbourhood of 0 in the box topology, but it cannot contain any of the open balls$B_d(0, y_n)$ a contradiction. Thus no such metric d exist and R^N under the box topology is not metrizable.

Definition: The Uniform Topology

Let X be a set, and Y be a metric space. For each $f \epsilon Y^X$and $\epsilon > 0$, define

$$B_\rho(f, \epsilon) = \{g \epsilon Y^X | \rho(f, g) < \epsilon\}$$

Then the sets $B_\rho(f, \epsilon)$ from a basis for a topology on Y^X, known as the uniform topology.

Example:- Consider the space $i^{¥}$, where i is given the standard metric. Let $f \epsilon i^{¥}$, be the constant zero function. Then the basic open set $B_r(f, 1)$ consist of all functions $g: ¥ \rightarrow i$ such that

$$Sup\{|g(k)||k \epsilon ¥\} < 1$$

Note: that $B_r(f, 1)$ is not simply box:

$$B_r(f, 1) \neq (-1,1) \times (-1,1) \times (-1,1) \times \ldots \ldots \ldots \ldots \ldots \ldots$$

The reason is that a function may take values in the interval (-1,1) but still have supremum equal to 1, for example

The function $g = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \ldots \ldots\right)$

lies in the box (-1,1) ¥, but $r(f, g) = 1$ and therefore $g \notin B_r(f, 1)$.
Incidentally, it can be shown that the box (-1, 1) ¥ is not even open in the uniform topology on$i^{¥}$ and hence uniform and box topologies are different on$i^{¥}$.

Theorem: Convergence in the Uniform Topology

Proof:-

Let x be a set and y b a metric space, let f_n be a sequence in Y^X and let $f \in Y^X$. Then $f_n \rightarrow f$under the uniform topology if and only if functions f_n converge uniformly to f.
Though, it may appear from the definition the uniform topology is a metric topology with metric r, this is not actually the case. The problem is that $r(f, g)$is often infinite, which is not allowed by the definition of a metric. This is less of a problem then it seems it works perfectly well to simply allow matrix to take infinite values. Alternatively we can define the bounded uniform metric $\bar{r}$ by

$$\bar{r}(f, g) = \min\{r(f, g), 1\}$$

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Then $\bar{r}$is a legitimate metric and the corresponding metric topology is the same as the uniform topology.

Theorem:$C(X,Y)$ is closed

Proof:-

Let X be a topological space, Y be a metric space and let

$$C(X,Y) = \{f: X \to Y | f \text{ is continous}\}$$

Then,$C(X,Y)$ is a closed subset of Y^X under the uniform topology.

Since Y^X is a metric space under the uniform topology this theorem is equivalent to the statement that the limit of any convergent sequence of points in $C(X,Y)$is an element of $C(X,Y)$ i.e. the uniform limit of a sequence of continuous function is again continuous.

Theorem: Comparing the three topologies

Let X be a set and Y be a metric space.

Let $T_{Product}$,T_{Box} and $T_{Uniform}$ denote the three topologies on Y^X. Then

$$T_{Product} \subset T_{Uniform} \subset T_{Box}$$

Proof:-We first prove that any sub basic open set in the product topology is open in the uniform topology. Let $S(x,U)$ be such a set, and let $f \in S(x,U)$.

Then U is open in Y and $f(x) \in U$, so there exists an $\in > 0$ such that $B(f(x), \in) \subset U$. Then every element of $B_\rho(f,\in)$ must also lie in $S(x,U)$. This proves that $S(x,U)$ is open in the uniform topology, and therefore $T_{Product} \subset T_{Uniform}$

Next we must show any basic open set in the uniform topology is open in the box topology. Let $B_\rho(f,\in)$ be such a set , and let $g \in B_\rho(f,\in)$. Then there exists and $\in^1 > 0$ so that $B_p(g,\in^1) \subset B_\rho(f,\in)$. Then $\pi_{x\in X}(g(x) - x, g(x) + \frac{\in^P}{2}$ isan open set in the box topology that contains g and it contained in $B_p(g,\in^{'})$, and is hence also contained in $B_p(f,\in)$.

This prove that $B_p(f,\in)$ is open in the box topology and therefore

$$T_{Uniform} \subset T_{Box}$$

Therefore,

$$T_{Product} \subset T_{Uniform} \subset T_{Box}$$

Conclusion:-

Note that the ordering of the three technologies above corresponds to how many sequence converge: lots of sequence converges in the product topology some sequence converge in the uniform topology and almost no sequence converges in the box topology.

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