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WEYL'S THEOREM FOR ALGEBRAICALLY $Q(A(k^*),m)$ OPERATORS

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ABSTRACT: In this piece of writing, we show that Weyl's theorem and property (ω) holds for algebraically $Q(A(k^*),m)$ operator.

KEYWORDS: SVEP, Isoloid, Weyl's theorem, $Q(A(k^*),m)$

1. INTRODUCTION AND PRELIMINARIES

Let H be an infinite dimensional Hilbert space $L(H)$ and denote the algebra of all bounded linear operator acting on H. Class $A(k)$ and absolute k -paranormal operator are studied detaille in [8]. Further as a generalization of Class $A(k)$, class $A(k^*)$ operator with $k \succ 0$ were studied in [6].

DEFINITION 1.1

Let $k \succ 0$ and m be a non negative integer. An operator $J \in L(H)$ is said to be an m -quasi class $A(k^*)$ operator (abbreviate as $Q(A(k^*),m)$) if $J^{*m}\left(J^*|J|^{2k}J\right)^{\frac{1}{k+1}}J^m \geq J^{*m}|J^*|^2 J^m$.

Lemma 1.2. [9]

Let $0<k<1$ and k be a non negative integer. If $J \in Q(A(k^*),m$ operator and $(J-\mu)x=0$ with $\mu \neq 0$, then $(J-\mu)^* x=0$.

Lemma 1.3

Let J be invertible and quasi nilpotent algebraically $Q(A(k^*),m)$ operator. Then J is nilpotent.

Proof: Suppose that $p(J)$ is $Q(A(k^*),m)$ operator for some non-constant polynomial p. Since $\sigma(p(J)) = p(\sigma(J))$, the operator $p(J) - p(0)$ is quasi-nilpotent, from Lemma 1.2 we have $CJ^m(J-\lambda_1)(J-\lambda_2)..............(J-\lambda_n) \equiv p(J) - p(0) = 0$ where $m \geq 1$. since $J - \lambda_i$ is invertible for every $\lambda_i \neq 0$ and so $J^m = 0$.

Lemma 1.4

Let J be an algebraically $Q(A(k^*),m)$ operator. Then J is isoloid.

Proof: Let $\lambda \in iso(\sigma(J))$ and let $E_\lambda = \frac{1}{2\pi i}\int_{\partial D\lambda}(Z-J)^{-1}dZ$ be the associated Riesz idempotent, where D_λ is a closed disc centered at λ which contains no other point of $\sigma(J)$. We can represent J as the direct sum, $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$ where $\sigma(J_1)=\lambda$ and $\sigma(J_2)=\sigma(J)\setminus\lambda$. Since J is algebraically $Q(A(k^*),m)$ operator $p(J)$ is a $Q(A(k^*),m)$ operator for some non-constant polynomial p. Since $\sigma(J_1)=\lambda$, we must have $\sigma(p(J_1)) = p(\sigma(J_1)) = p(\lambda)$. Therefore $p(J_1) = p(\lambda)$ is quasi-nilpotent.

Since $p(J_1)$ is $Q(A(k^*),m)$ operator, it follows from Lemma 1.2, that $p(J_1) = p(\lambda) = 0$. Put $q(z) = p(z) - p(\lambda)$. Then $q(J_1)=0$ and hence J_1 is algebraically $Q(A(k^*),m)$ operator. Since $J_1 - \lambda$ is quasi-nilpotent and algebraically $Q(A(k^*),m)$ operator, it follows from Lemma 1.3, then $J_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(J_1)$ and hence $\lambda \in \pi_0(J_1)$. This prove that J is isoloid.

Lemma 1.5

Let J be an algebraically $Q(A(k^*),m)$ operator. Then J is polaroid.

Theorem 1.6. [9]

Let J belongs to the $Q(A(k^*),m)$ operator. Then J is of finite ascent.

Corollary 1.7. [9]

Let J belongs to the $Q(A(k^*),m)$ operator. Then J has SVEP.

Theorem 1.8

Let J be an algebraically $Q(A(k^*),m)$ operator. Then J has SVEP.

Proof: First we show that if J is $Q(A(k^*),m)$ operator, then J has SVEP. Suppose that J is $Q(A(k^*),m)$ operator. If $\pi_0(J)=\phi$, then clearly J has SVEP. Suppose that $\pi_0(J)\neq\phi$, Let $\Delta(J) = \lambda \in \pi_0(J) : N(J-\lambda) \subseteq N(J^* - \overline{\lambda})$. Since J is $Q(A(k^*),m)$ operator and $\pi_0(J)\neq\phi$, $\Delta(J)\neq\phi$. Let M be the closed linear span of the subspaces $N(J-\lambda)$ with $\lambda\in\Delta(J)$. Then M reduces J and we can write J as $J_1 \oplus J_2$ on

$H = M \oplus M^{\perp}$. Clearly J_1 is normal and $\pi_0(J_2) = \phi$. Since J_1 and J_2 have both SVEP, J has SVEP. Suppose that J is algebraically $Q(A(k^*), m)$ operator. Then $p(J)$ is $Q(A(k^*), m)$ operator for some non constant polynomial p. Since $p(J)$ has SVEP, it follows from [12 ,Theorem 3.3.9] that J has SVEP.

2. WEYL'S THEOREM FOR ALGEBRAICALLY $Q(A(k^*), m)$ OPERATORS

Theorem 2.1.Let J be an algebraically $Q(A(k^*), m)$ operator. Then Weyl's theorem holds for J .

Proof: Suppose that $\lambda \in \sigma(J) \setminus w(J)$. Then $J - \lambda$ is Weyl and not invertible, we claim that $\lambda \in \partial\sigma(J)$. Assume that λ is an interior point of $\sigma(J)$. Then there exist a neighbourhood U of λ,such that $\dim N(J - \mu) > 0$ for all $\mu \in u$. It follows from [7,Theorem 10] that J does not have single valued extension property[SVEP]. On the other hand, since $p(J)$ is $Q(A(k^*), m)$ operator for some non constant polynomial p , it follows from Corollary 1.7. That J has SVEP. It is a contradiction, Therefore $\lambda \in \sigma(J) \setminus w(J)$ and it follows from the punctured neighbourhood theorem that $\lambda \in \pi_{00}(J)$.

Conversely suppose that $\lambda \in \pi_{00}(J)$.Using the Riesz idempotent $E_\lambda = \frac{1}{2\pi i}\int_{\partial D\lambda}(Z - J)^{-1}dZ$ for λ , we can represent J as the direct sum $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$ where $\sigma(J) = \lambda$ and $\sigma(J_2) = \sigma(J) \setminus \lambda$.

Now we consider two cases

case(i) $\lambda = 0$

Then J_1 is algebraically $Q(A(k^*), m)$ operator and quasi nilpotent. It follows from Lemma 1.3, that J_1 is nilpotent. We claim that $\dim R(E) < \infty$. For if $N(J_1)$ is infinite dimensional, then 0 does not belongs to $\pi_{00}(J)$. It is contradiction. Therefore J_1 is an operator on the finite dimensional space R(E) . So it follows that J_1 is Weyl. But since J_2 is invertible, we can conclude that J is Weyl. therefore $0 \in \sigma(J) \setminus w(J)$.

case(ii)

$\lambda \neq 0$.Then by Lemma 1.4, $J_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(J)$, $J_1 - \lambda$ is an operator on the finite dimensional space R(E) . So $J_1 - \lambda$ is Weyl. Since $J_2 - \lambda$ is invertible $J - \lambda$ is Weyl.

By case(i) and case(ii) , Weyl's theorem holds for J .

This complete the proof.

Theorem 2.2

Let J be an algebraically $Q(A(k^*), m)$ operator. Then Weyl's theorem holds for $f(J)$ for every $f \in H(\sigma(J))$.

Proof: Let $f \in H(\sigma(J))$.Since $w(f(J)) \subseteq f(w(J))$, it suffices to show that $f(w(J)) \subseteq w(f(J))$. Suppose $\lambda \notin w(f(J))$, then $f(J) - \lambda$ is Weyl and

$$f(J) - \lambda = C(J-\alpha_1)(J-\alpha_2)..................(J-\alpha_n)g(J)...........(I)$$

where $C, \alpha_1, \alpha_2, ...\alpha_n \in C$ and $g(J)$ is invertible. Since the operators in the right side (I) commute, every $J - \alpha_i$ is fredholm. Since J is algebraically $Q(A(k^*), m)$ operator. J has SVEP by Lemma 1.7. It follows from [1,Theorem 2.6] that $ind(J - \alpha_i) \leq 0$ for each i=1,2,3,......n . Therefore $\lambda \notin f(w(J))$ and hence $f(w(J)) \subseteq w(f(J))$.

Now by [13] , that is J is isoloid, then $f(\sigma(J) \setminus \pi_{00}(J)) = \sigma(f(J)) \setminus \pi_{00}(J)$ for every $f \in H(\sigma(J))$.

Since J is isoloid by Lemma 1.4 and Weyl's theorem holds for J by Theorem 2.1 $\sigma(f(J) \setminus \pi_{00}(J)) = f(\sigma(J) \setminus \pi_{00}(J)) = f(w(J)) = w(f(J))$ which implies that Weyl's theorem holds for $f(J)$. This complete the proof.

Theorem 2.3

Let J be an algebraically $Q(A(k^*), m)$ operator. Then generalized Weyl's theorem holds for J .

Proof: Assume that $\lambda \in \sigma(J) \setminus \sigma_{Bw}(J)$. Then $J - \lambda I$ is B- Weyl and not invertible. We claim that $\lambda \in \partial\sigma(J)$. Assume to the contrary that λ is an interior point of $\sigma(J)$. Then there exists a neighbor hood U of λ such that $\dim(J - \mu) > 0$ for all $\mu \in U$. It follows from [7, Theorem 10] that J doesnot have SVEP. On the other hand since $p(J)$ is $Q(A(k^*), m)$ operator for non-constant polynomial p . It follows from Corollary 1.7 that $p(J)$ has SVEP. Hence by [12, Theorem 3.3.9] J has SVEP, a contradiction.

Therefore $\lambda \in \partial\sigma(J)$. Conversely, assume that $\lambda \in E(J)$, then λ is isolated in $\sigma(J)$. From [10, Theorem 7.1] , we have $X = M \oplus N$, where M, N are closed subspaces of X , $U = (J - \lambda I)|_N$ is an invertible operator and $V = (A - \lambda I)|_N$ is a quasi - nilpotent operator. Since J is algebraically $Q(A(k^*), m)$ operator, V is also algebraically $Q(A(k^*), m)$ operator from Lemma 1.3, V is nilpotent. Therefore $J - \lambda I$ is Drazin invertible [5, Proposition 19] and [Corollary 2.2]. By [4 Lemma 4.1] $J - \lambda I$ is a B - fredholm operator of index 0.

Theorem .2.4

Assume that J or J^* is algebraically $Q(A(k^*), m)$. Then $\sigma_{ea}(f(J)) = f(\sigma_{ea}(J))$ for every $f \in H(\sigma(J))$.

Proof: Let $f \in H(\sigma(J))$. It suffices to that $f(\sigma_{ea}(J)) \subseteq \sigma_{ea}(f(J))$ for every $f \in H(\sigma(J))$. Suppose that $\lambda \in \sigma_{ea}(f(J))$ .Then

$$f(J) - \lambda = C(J - \alpha_1)(J - \alpha_2).................(J - \alpha_n)g(J)..........(I)$$

where $C, \alpha_1, \alpha_2, ...\alpha_n \in C$ and g(J) is invertible. If J is algebraically $Q(A(k^*), m)$ operator, it follows from [1, Theorem 2.6] that $i(J - \alpha_i) \leq 0$ for each i = 1,2,3,........n. Therefore λ does not belongs to $f(\sigma_{ea}(J))$, and hence $\sigma_{ea}(f(J)) = f(\sigma_{ea}(J))$. .

Suppose that J^* is algebraically $Q(A(k^*), m)$ then J^* is SVEP. Since $i(J - \alpha_i) \leq 0$ for each i=1,2,3....n. $(J - \alpha_i)$ is Weyl for each i = 1,2,3,.......n. Hence $\lambda \notin f(\sigma_{ea}(J))$, and so $\sigma_{ea}(f(J)) = f(\sigma_{ea}(J))$. .

This complete the proof.

3. Property (ω)

Definition 3.1

A bounded operator $J \in L(H)$ is said to satisfy propert ω if $E_0(J) = \Delta^a(J) = \sigma_a(J) \setminus \sigma_{SF_+^-}(J)$.

As observed in [3], we have either of a-Weyls theorem or property (ω) for J $\Rightarrow$ Weyl's theorem holds for J .

Lemma 3.2. [3]

Suppose that $J \in L(H)$.

(i) If J^* has the SVEP then $\sigma_{SF_+^-}(J) = \sigma_b(J)$.

(ii) If J has the SVEP then $\sigma_{SF_+^-}(J) = \sigma_b(J)$.

Theorem 3.3.

Let $J \in L(H)$.

(i) If J^* is algebraically $Q(A(k^*), m)$ then property (ω) holds for J.

(ii) If J is algebraically $Q(A(k^*), m)$ then property (ω) holds for J^*.

Proof: (i) Since J^* is algebraically of $Q(A(k^*), m)$, then J^* is the SVEP and J is polaroid by Lemma 1.5 because J is polaroid if and only if J^* is polaroid. Consequently $\sigma(J) = \sigma_a(J)$. If $iso\sigma(J) = \phi.$, then $E_0(J) = \phi.$. We show that $\sigma_a(J) \setminus \sigma_{SF_+^-}(J)$ is empty. By Lemma 3.2 we have $\sigma_a(J) \setminus \sigma_{SF_+^-}(J) = \sigma(J) \setminus \sigma_b(J)$ and the last set is empty, since $\sigma(J)$ has no isolated points. Therefore, J satisfies property (ω).

Consider the other case, $iso\sigma(J) \neq \phi.$ Suppose that $\lambda \in E_0(J)$. Then λ is isolated in $\sigma(J)$ and hence, by the polaroid condition, λ is a pole of the resolvent of J , i.e. $a(J - \lambda) = d(J - \lambda) < \infty$. By assumption $\alpha(J - \lambda) = < \infty$, so by [2,Theorem 3.1] $\beta(J - \lambda) = < \infty$, and hence $(J - \lambda)$ is a Fredholm operator. Therefore, by Lemma 3.2 $\lambda \in \sigma(J) \setminus \sigma_b(J) = \sigma_a(J) \setminus \sigma_{SF_+^-}(J)$. Conversely, if $\lambda \in \sigma_a(J) \setminus \sigma_{SF_+^-}(J) = \sigma(J) \setminus \sigma_b(J)$ then λ is an isolated point of $\sigma(J)$. Clearly, $0 < \alpha(J - \lambda) < \infty$, so $\lambda \in E_0(J)$ and hence J satisfies property (ω).

(ii) First note that since J has SVEP then $\sigma_a(J^*) = \{\lambda \in C : J - \lambda\, is\, not\, onto\} = \sigma(J) = \sigma(J^*)$. Suppose first that $iso\sigma(J) = iso\sigma(J^*)$. . Then $E_0(J^*) = \phi$. By Lemma 3.2 we have $\sigma_a(J) \setminus \sigma_{SF_+^-}(J) = \sigma(J) \setminus \sigma_b(J) = \phi$ so J^* satisfies property (ω).

Suppose that $iso\sigma(J) \neq \phi.$; and let $\lambda \in E_0(J^*)$. Then λ is isolated in $\sigma(J) = \sigma(J^*)$, hence a pole of the resolvent of J^* , since J^* is polaroid by Lemma 1.5. By assumption $\alpha(J^* - \overline{\lambda}) < \infty$ and since the ascent and the descent of $J^* - \overline{\lambda}$ are both finite it then follows by [2, Theorem 3.1 that $\alpha(J - \lambda) = \beta(J - \lambda) < \infty$, so $J^* - \overline{\lambda}$ is Browder and hence also $J - \lambda$ Browder. Therefore, $\lambda \in \sigma(J) \setminus \sigma_b(J)$ and by Lemma 3.2 it then follows that $\lambda \in \sigma_a(J) \setminus \sigma_{SF_+^-}(J)$.

Conversely, if $\lambda \in \sigma_a(J) \setminus \sigma_{SF_+^-}(J) = \sigma(J) \setminus \sigma_b(J)$, then λ is an isolated point of the spectrum of $\sigma(J) = \sigma(J^*)$. Hence $J - \lambda$ is Browder, or equivalently $J^* - \overline{\lambda}$ is Browder. Since $\alpha(J^* - \overline{\lambda}) = \beta(J^* - \overline{\lambda})$ we then have $\alpha(J^* - \overline{\lambda}) > 0$ .Clearly, $\alpha(J^* - \overline{\lambda}) < \infty$, since by assumption $J^* - \overline{\lambda} \in \sigma_{SF_+^-}(H)$ so that $\lambda \in E_0(J^*)$. Thus J^* satisfies property (ω) .

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