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ON A CLASS OF STARLIKE FUNCTIONS OF COMPLEX ORDER USING q-DIFFERENTIAL OPERATOR

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Abstract

A subclass of starlike functions of complex order is defined using q-differential operator which unifies well-known Dziok-Srivastava operator and Sălăgean differential operator. Coefficient inequalities, sufficient condition and an interesting subordination result are obtained. Finally, we give relevant connections of our main results with former results obtained by various other authors.

Keywords:
q-calculus;
univalent functions;
starlike functions;
convex function, subordination;
Dziok-Srivastava operator.

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1. Introduction

We let $\mathcal{A}$ to denote the class of all analytic function of the form

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{1.1}$$

in the open unit disc $\mathcal{U} = \{\ z: z \in \mathbb{C};\ |z| < 1\}$. Also we let $\mathcal{S}$ to denote the subclass of $\mathcal{A}$ which are analytic and univalent in $\mathcal{U}$. We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. For detailed study on the development of various studies on univalent function theory, we refer to [5, 8].

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

In 1908, Jackson [9] reintroduced the Euler-Jackson q-difference operator

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} (z \in \mathcal{U} - \{0\};\ q \in \mathbb{C}\backslash\{0\}),$$

where $\mathbb{C}$ denotes the set of complex numbers. The limit as q approaches 1^- is the derivative $\lim_{q\to 1} D_q f(z) = f'(z)$,provided the derivative exists. For example,

$$D_q(z^\alpha) = \frac{z^\alpha - (qz)^\alpha}{z(1-q)} = [\alpha]_q z^{\alpha-1}, \alpha \in \mathbb{C},$$

where

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, [0]_q = 0, \qquad q \in \mathbb{C}.$$

If $f(z)$ is of the form (1.1), a simple computation yields

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, (z \in \mathcal{U}), \tag{1.2}$$

and $D_q f(0) = f'(0)$, where $q \in (0,1)$. The application of q −calculus was initiated by Jackson [9, 10]. He was the first to develop the q −integral and q −derivative in a systematic way. Later, geometrical interpretation of the q −analysis has been recognized through studies on quantum groups. Simply, the quantum calculus is ordinary classical calculus without the notion of limits. It defines q −calculus and h −calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. For study on the development of q-claculus in slow motion, we refer to [6]. And a comprehensive study on the applications of q −calculus in the operator theory may be found in [2].

The q-hypergeometric series was developed by Heine as a generalization of the hypergeometric series

$$ {}_2F_1[a,b;\ c|q,z] = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n} z^n \tag{1.3} $$

where the q-shifted factorial is given by

$$ (a;q)_n = \begin{cases} 1, & \mathrm{n} = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & \mathrm{n} = 1,2,3,\ldots \end{cases} $$

and it is assumed that $c \neq q^{-m}$ for $m = 0,1,2,\ldots$. Generalizing the Heine's series, we define rϕ_s the basic hypergeometric series by

$$ {}_r\varphi_s = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\ldots(b_s;q)_n(q;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \tag{1.4} $$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $r > s+1$. In (1.3) and (1.4), it is assumed that the parameters $b_1, b_2, \ldots, b_s$ are such that the denominators factors in the terms of the series are never zero.

For complex parameters $a_1,\ldots,a_r$ and $b_1,\ldots,b_s\big(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- = 0,-1,-2,\ldots;\ j = 1,\ldots,s\big)$, we define the generalized q-hypergeometric function ${}_r\Psi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;\ q,z)$ by

$$ {}_r\Psi_s\big(a_1,a_2,\ldots,a_q;b_1,b_2,\ldots,b_s;\ q,z\big) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(q;q)_n(b_1;q)_n\ldots(b_s;q)_n} z^n \tag{1.5} $$

$$ (r = s+1;\ r,s \in \mathcal{N}_0 = \mathcal{N} \cup \{0\};\ z \in \mathcal{U}), $$

where $\mathcal{N}$ denotes the set of positive integers. By using the ratio test, we should note that, if $|q| < 1$, the series (1.5) converges absolutely for $|z| < 1$ and $r = s+1$. For more mathematical background of these functions, one may refer to [7].

Corresponding to a function $\mathcal{G}_{r,s}(a_i, b_j;\ q, z)(i = 1,2, \dots, r;\ j = 1,2, \dots, s)$ defined by

$$\mathcal{G}_{r,s}(a_i, b_j;\ q, z):$$
$$= z \quad {}_r\Psi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s;\ q, z) \tag{1.6}$$

We now define the following operator $\mathcal{J}_\lambda^m(a_1, b_1;\ q, z)f : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\mathcal{J}_\lambda^0(a_1, b_1;\ q, z)f(z) = f(z) * \mathcal{G}_{r,s}(a_i, b_j;\ q, z)$$

$$\mathcal{J}_\lambda^1(a_1, b_1;\ q, z)f(z)$$
$$= (1 - \lambda)\left(f(z) * \mathcal{G}_{r,s}(a_i, b_j;\ q, z)\right) + \lambda\, zD_q\left(f(z) * \mathcal{G}_{r,s}(a_i, b_j;\ q, z)\right) \tag{1.7}$$

$$\mathcal{J}_\lambda^m(a_1, b_1;\ q, z)f(z)$$
$$= \mathcal{J}_\lambda^1\left(\mathcal{J}_\lambda^{m-1}(a_1, b_1;\ q, z)f(z)\right) \tag{1.8}$$

If $f \in \mathcal{A}_1$, then from (1.7) and (1.8) we may easily deduce that

$$\mathcal{J}_\lambda^m(a_1, b_1;\ q, z)f$$
$$= z + \sum_{n=2}^{\infty}\left[1 - \lambda + [n]_q\lambda\right]^m \Upsilon_n c_n z^n \tag{1.9}$$
$$(m \in N_0 = N \cup \{0\}\ and\ \lambda \geq 0),$$

where

$$\Upsilon_n = \frac{(a_1; q)_{n-1}(a_2; q)_{n-1} \dots (a_r; q)_{n-1}}{(q; q)_{n-1}(b_1; q)_{n-1} \dots (b_s; q)_{n-1}}, (|q| < 1).$$

Remark 1.1 We note that the linear operator (1.9) is q-analogue of the operator defined by Selvaraj and Karthikeyan [12]. Here we list some special cases of the operator $\mathcal{J}_\lambda^m(a_1, b_1;\ q, z)f$.

1. For a choice of the parameter $m = 0$, the operator $\mathcal{J}_\lambda^0(\alpha_1, \beta_1)f(z)$ reduces to the q-analogue of Dziok- Srivastava operator [4].
2. For $a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, \dots, (i = 1, \dots, r, j = 1, \dots, s)$ and $q \to 1^-$, we get the operator defined by Selvaraj and Karthikeyan [12].

3. For $m=0, a_i=q^{\alpha_i}, b_j=q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0,-1,-2, \ldots,(i=1, \ldots, r, j=1, \ldots, s)$ and $q \rightarrow 1^{-}$, we get the well-known and famous Dziok-Srivastava operator.

4. For $r=2, s=1 ; a_1=b_1, a_2=q$ and $\lambda=1$, we get the q-analogue of the well knownSălăgean operator (see [11]).

Also many (well known and new) integral and differential operators can be obtained by specializing the parameters.

Using the operator $\mathcal{J}_\lambda^m(a_1, b_1 ; q, z) f$, we define $\mathcal{T}_\lambda^m(b ; a_1, b_1 ; q ; A, B)$ to be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$1+\frac{1}{b}\left(\frac{\mathcal{J}_\lambda^{m+1}(a_1, b_1 ; q, z) f}{\mathcal{J}_\lambda^m(a_1, b_1 ; q, z) f}-1\right) \prec \frac{1+A z}{1+B z}, \quad z \in \mathcal{U}, \qquad (1.10)$$

where $b \in \mathbb{C} \backslash\{0\}$, A and B are arbitrary fixed numbers,$-1 \leq B<A \leq 1$, $m \in \mathbb{N}_0$.

We note that by specializing $m, \lambda, r, s, a_1, b_1, A, B$ in the function class$\mathcal{T}_\lambda^m(b ; a_1, b_1 ; q ; A, B)$, we obtain several well-known and new subclasses of analytic functions. Here we list a few of them:

1. If we let $\lambda=1, r=2, s=1$, $a_1=b_1 a_2=q$ and $q \rightarrow 1^{-}$, then the class $\mathcal{T}_\lambda^m(b ; a_1, b_1 ; q ; A, B)$ reduces to the well- known class

$$\mathcal{H}^m(b ; A, B):=\{f: f \in \mathcal{A}, 1+\frac{1}{b}(\frac{\mathcal{D}^{m+1} f(z)}{\mathcal{D}^m f(z)}-1) \prec \frac{1+A z}{1+B z}, \qquad z \in \mathcal{U}\}$$

where $\mathcal{D}^m f$ is the well- known Sălăgean operator. The class $\mathcal{H}^m(\delta ; A, B)$ was introduced and studied by Attiya in [3].

2. For a choice of the parameter $\lambda=1,\ r=2, s=1,\ a_1=b_1, a_2=q,\ q \rightarrow 1^{-}$, $A=1$ and $B=-M$, the class $\mathcal{T}_\lambda^m(b ; a_1, b_1 ; q ; A, B)$ reduces to the class

$$\mathcal{H}^m(b ; M):=\{f: f \in \mathcal{A},\left|\frac{b-1+\frac{\mathcal{D}^{m+1} f(z)}{\mathcal{D}^m f(z)}}{b}-M\right|<M, z \in \mathcal{U}\}$$

where $M>\frac{1}{2}$. The class $\mathcal{H}^m(b ; M)$ was introduced and studied by Aouf, Darwish and Attiya in [1].

Apart from the above, several other well known and new classes of analytic functions can be obtained by specializing the parameters involved in the class $\mathcal{T}_\lambda^m(b ; a_1, b_1 ; q ; A, B)$.

2. Coefficient estimates

Theorem 2.1 *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{T}_\lambda^m(b;\ a_1,b_1;\ q;\ A,B)$ and let $\Gamma_n = \left|(A-B)b - B\lambda\left([n]_q - 1\right)\right| - \lambda\left([n]_q - 1\right)$.*

(a) If $\Gamma_2 \leq 0$, then

$$|c_j| \leq \frac{(A-B)|b|}{\left[1-\lambda+[j]_q\lambda\right]^m \lambda\left([j]_q - 1\right)\Upsilon_j}. \tag{2.1}$$

(b) If $\Gamma_n \geq 0$, then

$$|c_j| \leq \frac{1}{\left[1-\lambda+[j]_q\lambda\right]^m \lambda^{j-1}\Upsilon_j} \prod_{n=2}^{j} \frac{\left|(A-B)b - \left([n-1]_q - 1\right)B\right|}{\left([n]_q - 1\right)} \tag{2.2}$$

(c) If $\Gamma_k \geq 0$ and $\Gamma_{k+1} \leq 0$ for $k = 2,3,\dots,j-2$,

$$|c_j| \leq \frac{1}{\left[1-\lambda+[j]_q\lambda\right]^m \left([j]_q - 1\right)\lambda^{j-1}\Upsilon_j} \prod_{n=2}^{k+1} \frac{\left|(A-B)b - \left([n-1]_q - 1\right)B\right|}{\left([n]_q - 1\right)} \tag{2.3}$$

The bounds in (2.1) and (2.2) are sharp for all admissible A, B, $b \in \mathbb{C}\backslash\{0\}$ and for each j.

Proof. Since $f(z) \in \mathcal{T}_\lambda^m(b; a_1, b_1;\ q;\ A, B)$, the inequality (1.10) gives

$$\left|\mathcal{J}_\lambda^{m+1}(a_1,b_1;\ q,z)f - \mathcal{J}_\lambda^{m}(a_1,b_1;\ q,z)f\right|$$

$$= \left\{[(A-B)b + B]\mathcal{J}_\lambda^{m}(a_1,b_1;\ q,z)f - B\mathcal{J}_\lambda^{m+1}(a_1,b_1;\ q,z)f\right\}w(z). \tag{2.4}$$

Equation (2.4) may be written as

$$\sum_{n=2}^{\infty} \left[1-\lambda+[n]_q\lambda\right]^m \lambda\left([n]_q - 1\right)\Upsilon_n c_n z^n \tag{2.5}$$

$$= \{(A-B)bz + \sum_{n=2}^{\infty} \left[(A-B)b - B\lambda\left([n]_q - 1\right)\right]\left[1-\lambda+[n]_q\lambda\right]^m \Upsilon_n c_n z^n\}w(z).$$

Or equivalently

$$\sum_{n=2}^{j}\left[1-\lambda+[n]_q\lambda\right]^m\lambda\left([n]_q-1\right)\Upsilon_n c_n z^n+\sum_{n=j+1}^{\infty}d_n z^n$$
$$=\left\{(A-B)bz+\sum_{n=2}^{j-1}\left[(A-B)b-B\lambda\left([n]_q-1\right)\right]\left[1-\lambda+[n]_q\lambda\right]^m\Upsilon_n c_n z^n\right\}w(z), \quad (2.6)$$

for certain coefficients d_n.Explicitly $d_n=[1+(k-1)\lambda]^m\lambda(k-1)\Gamma_k a_k-[(A-B)b-B(k-2)\lambda][1+(k-2)\lambda]^m\Gamma_{k-1}a_{k-1}z^{-1}$.

Since $|w(z)|<1$, we have

$$\left|\sum_{n=2}^{j}\left[1-\lambda+[n]_q\lambda\right]^m\lambda\left([n]_q-1\right)\Upsilon_n c_n z^n+\sum_{n=j+1}^{\infty}d_n z^n\right|$$
$$\leq\left|(A-B)bz+\sum_{n=2}^{j-1}\left[(A-B)b-B\lambda\left([n]_q-1\right)\right]\left[1-\lambda+[n]_q\lambda\right]^m\Upsilon_n c_n z^n\right|. \quad (2.7)$$

Let $z=re^{i\theta}, r<1$, applying the Parseval's formula (see [5] p.138) on both sides of the above inequality and after simple computation , we get

$$\sum_{n=2}^{j}\left[1-\lambda+[n]_q\lambda\right]^{2m}\lambda^2\left([n]_q-1\right)^2\Upsilon_n^2|c_n|^2r^{2n}+\sum_{n=j+1}^{\infty}|d_n|^2r^{2n}$$
$$\leq(A-B)^2|b|^2r^2+\sum_{n=2}^{j-1}\left|(A-B)b-B\lambda\left([n]_q-1\right)\right|^2\left[1-\lambda+[n]_q\lambda\right]^{2m}\Upsilon_n^2|c_n|^2r^{2n}.$$

Let $r\longrightarrow 1^-$, then on some simplification we obtain

$$\left[1-\lambda+[j]_q\lambda\right]^{2m}\lambda^2\left([j]_q-1\right)^2\Upsilon_j^2|c_j|^2 \leq(A-B)^2|b|^2$$
$$+\sum_{n=2}^{j-1}\left\{\left|(A-B)b-B\lambda\left([n]_q-1\right)\right|^2-\lambda^2\left([n]_q-1\right)^2\right\}\left[1-\lambda+[n]_q\lambda\right]^{2m}\Upsilon_n^2|c_n|^2 \quad j\geq 2. \quad (2.8)$$

For $j=2$, it follows from (2.8) that

$$|c_2|\leq\frac{(A-B)|b|}{\left[1-\lambda+[2]_q\lambda\right]^m\lambda\left([2]_q-1\right)\Upsilon_2}$$
$$=\frac{(A-B)|b|}{[1+q\lambda]^m q\lambda\Upsilon_2}. \quad (2.9)$$

Since

$$\left|(A-B)b-B\lambda[n-1]_q-1\right| \geq \left|(A-B)b-B\lambda[n]_q-1\right|-|B| \geq [n]_q-2,$$

if $\Lambda_n \geq 0$ then $\Lambda_{n-1} \geq 0$ for $n = 2,3,\ldots$. Again, if $\Lambda_n \leq 0$ then $\Lambda_{n+1} \leq 0$ for $n = 2,3,\ldots$, because

$$\left|(A-B)b-B\lambda[n+1]_q-1\right| \leq \left|(A-B)b-B\lambda[n]_q-1\right|+|B| \geq [n]_q.$$

If $\Lambda_2 \leq 0$, then from the above discussion we can conclude that $\Lambda_n \leq 0$ for all $n > 2$. It follows from (2.8) that

$$|c_j| \leq \frac{(A-B)|b|}{\left[1-\lambda+[j]_q\lambda\right]^m \lambda\left([j]_q-1\right)\Upsilon_j}. \tag{2.10}$$

If $\Gamma_{n-1} \geq 0$, then from the above observation, $\Gamma_2, \Gamma_3, \ldots, \Gamma_{j-2} \geq 0$. From (2.10), we infer that the inequality (2.2) is true for $j = 2$. We establish (2.2), by mathematical induction. Suppose (2.2) is valid for $n = 2,3,\ldots,(j-1)$. Then it follows from (2.8) that

$$\begin{aligned}
&\left[1-\lambda+[j]_q\lambda\right]^{2m}\lambda^2\left([j]_q-1\right)^2\Upsilon_j^2|c_j|^2 \\
&\qquad \leq (A-B)^2|b|^2 \\
&\qquad\qquad + \sum_{n=2}^{j-1}\left\{\left|(A-B)b-B\lambda\left([n]_q-1\right)\right|^2\right. \\
&\qquad\qquad \left. -\lambda^2\left([n]_q-1\right)^2\right\}\left[1-\lambda+ [n]_q\lambda\right]^{2m}\Upsilon_n^2|c_n|^2
\end{aligned}$$

$$\begin{aligned}
&\leq (A-B)^2|b|^2 + \sum_{n=2}^{j-1}\left\{|(A-B)b-B\lambda\left([n]_q-1\right)]^2-\lambda^2\left([n]_q-1\right)^2\right\}\left[1-\lambda+ [n]_q\lambda\right]^{2m}\Upsilon_n^2 \\
&\qquad \times\left\{\frac{1}{\left[1-\lambda+[n]_q\lambda\right]^{2m}\Upsilon_n^2\{\lambda^{n-1}\}^2}\prod_{j=2}^{n}\frac{|(A-B)b-([k-1]_q-1)B|^2}{\left([n]_q-1\right)^2}\right\}
\end{aligned}$$

Thus, we get

$$|c_j| \leq \frac{1}{\left[1-\lambda+[j]_q\lambda\right]^m\lambda^{j-1}\Upsilon_j}\prod_{n=2}^{j}\frac{|(A-B)b-\left([n-1]_q-1\right)B|}{\left([n]_q-1\right)},$$

which completes the proof of (2.2).

Now if we assume that $\Gamma_n \geq 0$ and $\Gamma_{n+1} \leq 0$ for $n = 2,3,\ldots,j-2$. Then $\Gamma_2, \Gamma_3, \ldots, \Gamma_{n-1} \geq 0$ and $\Gamma_{n+2}, \Gamma_{n+3}, \ldots, \Gamma_{j-2} \leq 0$. Then (2.8) gives

$$\left[1-\lambda+[j]_q\lambda\right]^{2m}\lambda^2\left([j]_q-1\right)^2\Upsilon_j^2\left|c_j\right|^2$$
$$\leq (A-B)^2|b|^2+\sum_{n=2}^{l}\left\{\left|(A-B)b-B\lambda\left([n]_q-1\right)\right|^2-\lambda^2\left([n]_q-1\right)^2\right\}\left[1-\lambda+[n]_q\lambda\right]^{2m}\Upsilon_n^2|c_n|^2$$
$$+\sum_{k=l+1}^{j-1}\left\{\left[(A-B)b-B\lambda\left([n]_q-1\right)\right]^2-\lambda^2\left([n]_q-1\right)^2\right\}\left[1-\lambda+[n]_q\lambda\right]^{2m}\Upsilon_n^2|c_n|^2$$
$$\leq (A-B)^2|b|^2+\sum_{n=2}^{l}\left[(A-B)b-B\lambda\left([n]_q-1\right)\right]^2\left[1-\lambda+[n]_q\lambda\right]^{2m}\Upsilon_n^2|c_n|^2.$$

On substituting upper estimates for $a_2, a_3, \dots, a_l$ obtained above and simplifying, we obtain (2.3).Also, the bounds in (2.1) are sharp for the functions $f_k(z)$ given by

$$\mathcal{J}_\lambda^m(a_1,b_1;\ q,z)f_k(z)=\begin{cases} z(1+Bz)^{\frac{(A-B)b}{B\lambda(k-1)}} & if\ B\neq 0,\\ z\ exp\left(\dfrac{Ab}{\lambda(k-1)}z^{k-1}\right) & if\ B=0\end{cases}$$

The bounds in (2.2) are sharp for the functions $f(z)$ given by

$$\mathcal{J}_\lambda^m(a_1,b_1;\ q,z)f=\begin{cases} z(1+Bz)^{\frac{(A-B)b}{B}} & if\ B\neq 0\\ z\ exp(Abz) & if\ B=0\end{cases}.$$

Remark 2.1 *Putting* $q\longrightarrow 1^-,\ r=2, s=1;\ a_1=b_1, a_2=q$ *and* $\lambda=1$ *in Theorem 2.1, we get the*

result due to Attiya [3].

For a choice of the parameters $a_i=q^{\alpha_i}, b_j=q^{\beta_j}, \alpha_i, \beta_j\in\mathbb{C}, \beta_j\neq 0,-1,-2,\dots,(i=1,\dots,r, j=1,\dots,s)$ and $q\to 1^-$, Theorem 2.1 reduces to

Corollary 2.2 *[13] Let the function* $f(z)$ *defined by (1.1) be in the class* $\mathcal{T}_\lambda^m(b;\ a_1,b_1;\ q;\ A,B)$. *Let* $\Lambda_n=|(A-B)b-B\lambda(n-1)|-\lambda(n-1)$ *and also let*

$$\Psi_k=\frac{(\alpha_1)_{k-1}\ \dots\ (\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ \dots\ (\beta_s)_{k-1}(k-1)!}.$$

(a) If $\Lambda_2\leq 0$, then

$$\left|c_j\right|\leq\frac{(A-B)|b|}{[1+(j-1)\lambda]^m\lambda(j-1)\Psi_j}. \tag{2.11}$$

(b) If $\Lambda_n \geq 0$, then

$$|c_j| \leq \frac{1}{[1+(j-1)\lambda]^m (j-1)!\,\lambda^{j-1}\Psi_j} \prod_{n=2}^{j} |(A-B)b-(n-2)B| \qquad (2.12)$$

(c) If $\Lambda_k \geq 0$ and $\Gamma_{k+1} \leq 0$ for $k = 2,3,\dots,j-2$,

$$|c_j| \leq \frac{1}{[1+(j-1)\lambda]^m (k-1)!\,(j-1)\lambda^{j-1}\Psi_j} \prod_{n=2}^{k+1} |(A-B)b-(n-2)B| \qquad (2.13)$$

The bounds in (2.11) and (2.12) are sharp for all admissible $A, B,\, b \in \mathbb{C}\backslash\{0\}$ and for each j.

If we let $\lambda = 1, r = 2, s = 1$, $a_1 = b_1$ and $\alpha_2 = q$, $A = 1$ and $B = -M$ in Theorem 2.1, we have

Corollary 2.3 *[1] Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{H}^m(b;M)$. Let*

$$G = \left[\frac{2u(n-1)\,Re\,(b)}{(n-1)^2(1-u)-|b|^2(1+u)}\right],$$

$$(for\ n = 1,3,\dots,j-1).$$

(a) If $2u(n-1)Re\,\{b\} > (n-1)^2(1-u) - |b|^2(1+u)$, then, for $j = 2,3,\dots,G+2$

$$|a_j| \leq \frac{1}{j^m(j-1)!} \prod_{n=2}^{j} |(1+u)b+(n-2)u| \qquad (2.14)$$

and for $j > G+2$

$$|a_j| \leq \frac{1}{j^m(j-1)(G+1)!} \prod_{n=2}^{G+3} |(1+u)b+(n-2)u|$$

(b) If $2u(n-1)\,Re\,\{b\} \leq (n-1)^2(1-u) - |b|^2(1+u)$, then

$$|a_j| \leq \frac{(1+u)|b|}{(j-1)j^m} \quad j \geq 2. \qquad (2.15)$$

where $u = 1 - 1M$ $(M > -12)$.The inequalities (2.14) and (2.15) are sharp.

3. A sufficient condition for a function to be in $\mathcal{T}_{\lambda}^{m}(b;a_1,b_1;q;A,B)$

Theorem 3.1 *Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m\{\lambda([n]_q-1)+|(A-B)b-B([n]_q-1)\lambda|\}\Upsilon_n|c_n|$$
$$\leq (A-B)|b| \qquad (3.1)$$

holds, then $f(z)$ belongs to $\mathcal{T}_{\lambda}^{m}(b;a_1,b_1;\ q;\ A,B)$.

Proof. Suppose that the inequality holds. Then we have for $z\in\mathcal{U}$

$$\left|\mathcal{J}_{\lambda}^{m+1}(a_1,b_1;\ q,z)f-\mathcal{J}_{\lambda}^{m}(a_1,b_1;\ q,z)f\right|-|(A-B)b\mathcal{J}_{\lambda}^{m}(a_1,b_1;\ q,z)f-$$
$$B\left[\mathcal{J}_{\lambda}^{m+1}(a_1,b_1;\ q,z)f-\mathcal{J}_{\lambda}^{m}(a_1,b_1;\ q,z)f\right]|$$
$$=|\sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m\lambda([n]_q-1)\Upsilon_n c_n z^n|-|(A-B)b[z$$
$$+\sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m\Upsilon_n c_n z^n]$$
$$-\ B\sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m\lambda([n]_q-1)\Upsilon_n c_n z^n|$$
$$\leq\ \sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m\{\lambda([n]_q-1)+|(A-B)b-B([n]_q-1)\lambda|\}\Upsilon_n|c_n|r^n-(A-B)|b|r.$$

Letting $r\longrightarrow 1^-$, then we have

$$\left|\mathcal{J}_{\lambda}^{m+1}(a_1,b_1;\ q,z)f-\mathcal{J}_{\lambda}^{m}(a_1,b_1;\ q,z)f\right|-|(A-B)b\mathcal{J}_{\lambda}^{m}(a_1,b_1;\ q,z)f-$$
$$B\left[\mathcal{J}_{\lambda}^{m+1}(a_1,b_1;\ q,z)f-\mathcal{J}_{\lambda}^{m}(a_1,b_1;\ q,z)f\right]|$$
$$\leq\sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m\{\lambda([n]_q-1)+|(A-B)b-B([n]_q-1)\lambda|\}\Upsilon_n|c_n|-(A-B)|b|$$
$$\leq 0.$$

Hence it follows that

$$\frac{\left|\frac{\mathcal{J}_{\lambda}^{m+1}(a_1,b_1;q,z)f}{\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f}-1\right|}{\left|B\left[\frac{\mathcal{J}_{\lambda}^{m+1}(a_1,b_1;q,z)f}{\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f}-1\right]-(A-B)b\right|}<1,\quad z\in\mathcal{U}.$$

Letting

$$w(z)=\frac{\frac{\mathcal{J}_\lambda^{m+1}(a_1,b_1;\,q,z)f}{\mathcal{J}_\lambda^{m}(a_1,b_1;\,q,z)f}-1}{B[\frac{\mathcal{J}_\lambda^{m+1}(a_1,b_1;\,q,z)f}{\mathcal{J}_\lambda^{m}(a_1,b_1;\,q,z)f}-1]-(A-B)b},$$

then $w(0)=0$, $w(z)$ is analytic in $|z|<1$ and $|w(z)|<1$. Hence we have

$$\frac{\mathcal{J}_\lambda^{m+1}(a_1,b_1;\ q,z)f}{\mathcal{J}_\lambda^{m}(a_1,b_1;\ q,z)f}=\frac{1+[B+b(A-B)]w(z)}{1+Bw(z)}$$

which shows that $f(z)$ belongs to $\mathcal{T}_\lambda^m(b;a_1,b_1;\ q;\ A,B)$.

4. Subordination Results for the Class $\mathcal{T}_\lambda^m(b;a_1,b_1;\ q;\ A,B)$

Definition 4.1 *A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence*

if, whenever $f(z)$ is analytic, univalent and convex in $\mathcal{U}$, we have the subordination given by

$$\sum_{k=1}^{\infty}b_k a_k z^k\prec f(z)(z\in\mathcal{U},a_1=1).\qquad(4.1)$$

Lemma 4.1 *[14]The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1+2\sum_{k=1}^{\infty}b_k z^k\right\}>0(z\in\mathcal{U}).\qquad(4.2)$$

For convenience, we shall henceforth denote

$$\sigma_k^q(b,\lambda,m,a_1,b_1,A,B)\qquad(4.3)$$
$$=\left[1-\lambda+[n]_q\lambda\right]^m\{\lambda([n]_q-1)+|(A-B)b$$
$$-B([n]_q-1)\lambda|\}\frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\dots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\dots(b_s;q)_{n-1}}.$$

Let $\tilde{\mathcal{T}}_\lambda^m(b;a_1,b_1;\ q;\ A,B)$ denote the class of functions $f(z)\in\mathcal{A}$ whose coefficients satisfy the conditions (3.1). We note that $\tilde{\mathcal{T}}_\lambda^m(b;a_1,b_1;\ q;\ A,B)\subseteq\mathcal{T}_\lambda^m(b;a_1,b_1;\ q;\ A,B)$.

Theorem 4.2 *Let the function $f(z)$ defined by (1.1) be in the class $\tilde{\mathcal{T}}_\lambda^m(b;\ a_1,b_1;\ q;\ A,B)$ where $-1\le B<A\le 1$. Also let $\mathcal{C}$ denote the familiar class of functions $f(z)\in\mathcal{A}$ which are also*

univalent and convex in $\mathcal{U}$*. Then*

$$\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}(f*g)(z)\prec g(z)(z\in\mathcal{U};\ g\in\mathcal{C}), \qquad (4.4)$$

and

$$\Re\mathrm{e}\ \left(f(z)\right)>-\frac{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}(z\in\mathcal{U}). \qquad (4.5)$$

The constant $\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}$ is the best estimate.

Proof. Let $f(z)\in\widetilde{\mathcal{H}}_\lambda^m(b;\alpha_1,\beta_1;A,B)$ and let $g(z)=z+\sum_{k=2}^{\infty}b_kz^k\in\mathcal{C}$. Then

$$\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}(f*g)(z)$$

$$=\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}(z+\sum_{k=2}^{\infty}a_kb_kz^k).$$

Thus, by Definition 4.1, the assertion of the theorem will hold if the sequence

$$\left\{\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence,with $a_1=1$. In view of Lemma 4.1, this will be true if and only if

$$\Re\mathrm{e}\ \left\{1+2\sum_{k=1}^{\infty}\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}a_kz^k\right\}>0 \quad (z\in\mathcal{U}). \qquad (4.6)$$

Now

$$\Re\mathrm{e}\ \{1+\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}\sum_{k=1}^{\infty}a_kz^k\}$$

$$=\Re\mathrm{e}\ \left\{1+\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}a_1z\right.$$
$$\left.+\frac{1}{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}\sum_{k=2}^{\infty}\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)a_kz^k\right\}$$

$$\geq 1-\left\{\left|\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}\right|r+\right.$$

$$\left.\frac{1}{|(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)|}\sum_{k=2}^{\infty}\sigma_k(b,\lambda,m,\alpha_1,\beta_1,A,B)|a_k|r^k\right\}.$$

Since $\sigma_k^q(b,\lambda,m,a_1,b_1,A,B)$ is a real increasing function of k $(k\geq 2)$

$$1-\left\{\left|\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}\right|r+\right.$$

$$\left.\frac{1}{|(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)|}\sum_{k=2}^{\infty}\sigma_k(b,\lambda,m,\alpha_1,\beta_1,A,B)|a_k|r^k\right\}$$

$$>1-\left\{\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}r\right.$$

$$\left.+\frac{(A-B)|b|}{(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}r\right\}=1-r>0.$$

Thus (4.6) holds true in $\mathcal{U}$. This proves the inequality (4.4). The inequality (4.5) follows by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$ in (4.4). To prove the sharpness of the constant $\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}$, we consider $f_0(z)\in\widetilde{\mathcal{H}}_\lambda^m(b;\alpha_1,\beta_1;A,B)$ given by

$$f_0(z)=z-\frac{(A-B)|b|}{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}z^2(-1\leq B<A\leq 1).$$

Thus from (4.4), we have

$$\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}f_0(z)\prec\frac{z}{1-z}.$$

(4.7)

It can be easily verified that

$$\min\left\{Re\left(\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}f_0(z)\right)\right\}=-\frac{1}{2}(z\in\mathcal{U}),$$

This shows that the constant $\frac{\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)}{2[(A-B)|b|+\sigma_2^q(b,\lambda,m,a_1,b_1,A,B)]}$ is best possible.

Remark 4.1 *By specializing the parameters, the above result reduces to various other results obtained by several authors.*

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