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Role of Mittag-Leffler Function in Susceptible InfectiousRecovered (SIR) Model

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Abstract: The purpose of this paper is to illuminate the distinguished role of the Mittag-Leffler function and its generalizations in fractional analysis and fractional Stochasticmodels. The content ofthepaperis history, properties and applications of Mittag-Leffler function connected to the SIR model.

Keywords: Mittag-Leffler function; fractional integrals and derivatives; fractional equations; fractional Stochastic models,continuous-time random walk (CTRW),SIR models etc.

Introduction:

In the paper, we would like to highlight the distinguished role of the Mittag-Leffler function and its numerous generalizations infractional calculus and fractional modeling. Partly,thematerialof the paper is based on the results from the recent monograph

$$E_\alpha(Z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad \mathrm{Re}\,(\alpha) > 0 \qquad (1)$$

The Mittag-Leffler function has been introduced to give an answer to a classical question of complex analysis, namely to describe the procedure of the analytic continuation of power series outside the disc of their convergence. Much later, theoretical applications to the study of integral equations, as well as more practical applications to the modeling of 'non-standard' processes have been found. The importance of the Mittag-Leffler function was re-discovered when its connection to fractional calculus was fully understood.Different aspects of the distinguished role of this function in fractional theory and its applications have been described in several monographs and surveys on fractional calculus and Fractional Modeling.

The paper is organized as follows. In Section 2, we briefly describe the history of the Mittag-Leffler function, paying attention mostly to the first period of the development of its theory. Section 3 is devoted to the presentation of the properties of the Mittag-Leffler function and its direct generalizations. Theoretical applications of the Mittag-Leffler function are briefly described in Section 4. Mostly, we deal here with applications to the study of integral and differential equations of fractional order. Lastly, Section 5 is devoted to the description of a number of Stochastic models in physics, mechanics, chemistry and biology in which the Mittag-Leffler function plays a crucial role. We also mention there the applications of the Mittag-Leffler function in probability theory.We have to point out that it is practically impossible to touch on all of the aspects of the theory of the Mittag-Leffler function and its generalizations, as well as their applications. The list of references is also incomplete. The interested reader can find more information in the monographs and surveys mentioned above (see also thereferencestherein).The most extended source for such information is the recent monograph

1. History of the Mittag-LefflerFunction:

At the end of the 19th century, Gösta Magnus Mittag-Leffler started to work on the problem of the analytic continuation of monogenic functions of one complex variable. This classical questionattractedtheattentionofthegreatmathematiciansofthattime.Inparticular,tosolvethe above problem, the construction related to the so-called Laplace–Abel integral wasproposed:

$$\int_0^\infty e^{-\omega} F(\omega z) d\omega \quad (2)$$

$$where F(z) = \sum_{\vartheta=0}^{\infty} \frac{n_\vartheta}{\vartheta!} z^\vartheta , \qquad \lim_{\vartheta \to \infty} Sup \sqrt[\vartheta]{|n_\vartheta|} = \frac{1}{r}$$

On the basis of his first results in the area, Mittag-Leffler made three reports in 1898 at the Royal Swedish Academy of Sciences in Stockholm. In particular, he proposed using the followinggeneralization of the Laplace–Abel integral:

$$\int_0^\infty e^{-\omega} E_\alpha(\omega^\alpha z) d\omega \qquad (3)$$

with$E_\alpha$defined by Equation (1). The properties of the latter were studied by him in a series of fivenotes published in 1901–1905 .Nowadays, the function E_α is known as theMittag-Leffler function.

Practically at the same time, several other functions related to the problem studied by Mittag-Lefflerwere introduced.

Among them are the functions introduced by Le Roy:

$$\sum_{k=0}^{\infty} \frac{z^k}{(k!)^p} , \quad p > 0 \qquad (4)$$

byLindelöf:

$$\sum_{k=0}^{\infty} \frac{z^k}{n^{\alpha k}} , \quad 0 < \alpha < 1 \qquad (5)$$

$$\sum_{k=0}^{\infty}\left(\frac{z}{\log\left(k+\frac{1}{\alpha}\right)}\right)^{k}, \quad 0<\alpha<1 \qquad (6)$$

and byMalmquist:

$$\sum_{n=0}^{\infty}\frac{z^{k-2}}{\Gamma\left(1+\frac{k}{(\log n)^{\alpha}}\right)}, \quad 0<\alpha<1 \quad (7)$$

The direct generalization of the Mittag-Leffler function into two parameters was proposed by Wiman in his work on zeros of function Equation (1):

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \text{Re}(\alpha)>0, \quad \beta\in\mathbb{C} \ (8)$$

Later,this function was rediscovered and intensively studied byAgarvalandHumbert. Withβ = 1, this function coincides with the classical Mittag-Leffler function,$E_{\alpha,1}(z)=E_{\alpha}(z)$. The most popular and widely-applicable functions from the above-mentioned collection are the Mittag-LefflerfunctionofoneparameterEquationandtheMittag-Lefflerfunctionoftwoparameter EquationThe two-parameter functionof the Mittag-Leffler type, which plays very important rolein the fractional calculus, was first introduced by Agarwal. A number of relationships for this function were obtained byHumbert and Agarwal using Laplace transform technique.In the year 1930, Hille and Tamarkin have obtain a solutionof the Abel-Voltera type equation in terms of Mittag-Lefflerfunction. Furthermore, the various properties, generalizationand applications of Mittag-Leffler function are studied by manyresearchers and Mathematician such as M.M. Dzhrbashyan ,Blair (1974), Bagley and Torvik (1984), Kilbas and Saigo (1995),Gorenflo et al. (1997), Gorenflo, Luchko and Rogosin (1997), Kilbas, Saigo and Saxena (2004), Saxena and Kalla(2008) etc. and contributed. In this connection, we prove somerelations of Mittag-Leffler functions of one and two parameters.

2. Properties of Mittag-Leffler Functions to other Functions:

1) $E_0(z) = \frac{1}{1-z}$, $|z| < 1$.
2) $E_1(z) = e^z$
3) $E_2(z) = \cosh(\sqrt{z})$ $\quad z\in C$
4) $E_{1,1}(z) = e^z$
5) $E_{1,2}(z) = \frac{e^z-1}{z}$ –
6) $E_{2,1}(z) = \cosh(\sqrt{z})$ $\quad z\in C$
7) $E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{(\sqrt{z})}$
8) $E_{\frac{1}{2},1}(z) = e^{z^2}$ er fc(-z) where, er fc(-z) is the complimentary error function

The next period in the development of the theory of the Mittag-Leffler function is connected with increasing the number of parameters. Thus, the three-parametric Mittag-Leffler-type function was introduced by Prabhakar.

$$E^{\gamma}_{\alpha,\beta}(z):=\sum_{k=0}^{\infty}\frac{(\gamma)_k z^k}{k!\,\Gamma(\alpha k+\beta)}, \quad \text{Re}(\alpha)>0, \quad \text{Re}(\beta)>0, \quad \gamma>0 \qquad (9)$$

where $(\gamma)_n = \gamma(\gamma + 1) \ldots (\gamma + n - 1)$. For $\gamma = 1$, this function coincides with two-parametric Mittag-Leffler function Equation (8), and with $\beta = \gamma = 1$, it coincides with the classical Mittag-Leffler function Equation (1), *i.e.*, $E_{\alpha,\beta,1} = E_{\alpha,\beta}$, $E_{\alpha,1,1} = E_{\alpha}$.

Later,inrelationtothesolutionofacertaintypeoffractionaldifferentialequations,Kilbasand Saigo introduced another kind of three-parametric Mittag-Leffler-typefunction:

$$E_{\propto,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k \ , \quad \text{Re}(\alpha) > 0, \ m > 0, \ l \in \mathbb{C} \ (10)$$

With $\propto, m, l$ are Parameters, for m=1 this function is reduced to two parametric function.Equations (9) and (10)) are essentially used as an explicitrepresentation of solutions to integral and differential equations of the fractional order.

The properties that we discuss here are the following:

The order ρ of an entire function $f(z)=\sum_{n=0}^{\infty} c_n z^n$,

$$\rho = \lim_{n\to\infty} Sup \frac{n \log n}{\log\frac{1}{|c_k|}}$$

The order σ of an entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$of the order ρ;

$$(\rho e \sigma)^{\frac{1}{\rho}} = \lim_{n\to\infty} Sup(n^{\frac{1}{\rho}} \sqrt[n]{|c_k|})$$

The Laplace transform of a function $f(t) = L\{f(t)\} = \emptyset(s) = \int_0^{\infty} e^{-st} f(t) dt$

The Fractional Integral of a function $f(x) = J_0^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \ x > 0$

The Fractional Derivative of a function $f(x) = D_0^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x \frac{f(t)}{(x-t)^{1-m+\alpha}} dt$, $x > 0$

$m - 1 < \propto < m$

Properties of the functions $E_{\alpha}(Z)$, $E_{\propto,\beta}(z)$,$E^{\gamma}_{\alpha,\beta}(z)$ and $E_{\propto,m,l}(z)$ are also discussed.

Several other analytic properties are discussed, in particular different types of recurrence relations, integral and differential properties. The study of zeros' distribution is presented there, too. These results are helpful for the investigation of certain inverse scattering problems, as well as other problems from the operator theory. For calculation of the Mittag-Leffler function, one can use. Asymptotic properties of the Mittag-Leffler-type functions show their place in the whole theory of special functions. Complete monotonicity of the Mittag-Leffler function is applied to the study of the Lévy stable distributions; relations to different kinds integral transforms are used as the solution to integral and differential equations.

3. Applications to Fractional Order Equations:

We obtained the solutions of various fractional order differentialequations and these are represented graphically by using Mathematica.The Abel integral equations occur in many situations where physical measurements are to be evaluated. Inmanyof these cases, the independent variable is the radius of a circle or a sphere, and only after a change of variables, the integral operator has the formI^{α},usuallywithα=1/2,andtheequation is of the first kind. For instance, there are applications in the evaluation of spectroscopicmeasurements of cylindrical gas discharges, the study of the solar or a planetary atmosphere, the investigation of star densities in a globular cluster, the inversion of travel times of seismic waves for the determination of terrestrialsub-surfacestructureandinthesolutionofproblemsinsphericalstereology.Descriptionsand analysis of several problems of this kind can be found in the books by Gorenflo andVessella.

Examples of initial value problem are easily solved by using laplace transform and Inverse laplace transform. Oneofthefirstinvestigationsofdifferentialequationsoffractionalorderwasmadeby Barrett. He considered differential equations with the fractional derivative of the Riemann–Liouville-type of arbitrary order α, Re α >0, where *n* boundary conditions (n = Re α+ 1) in the form of the values at the initial point of the fractional derivatives of order$\alpha - k$,k=1,2,...,$n$areposed.Itwasshownthat inasuitable class of functions, the solution is uniqueandisrepresentedusingtheMittag-Lefflerfunction.

One of the leading methods for linear fractional ordinary differential equations is their (equivalent) reduction to certain Volter raintegralequations in proper functional spaces.An extended technique based on this method in the application to different kinds of fractionaldifferentialequationsispresented. Other methods proposed for explicitly solving ordinary fractional differential equations and discussed are the compositional method, the operational method and the integral transforms method.

It was noted, in particular, that different types of fractional derivatives involved in the equationslead to different kinds of initial conditions, e.g., if a differential equation contains the Riemann–Liouville fractional derivative, then the natural initial conditions are so-called Cauchy-type conditions, *i.e.*, conditions of the type, but in the case of the Caputo derivatives, it is natural to pose the standard Cauchyconditions.

$$D^{\beta_n}_{a+}(a+) = b_n$$

It was probably Dzherbashian who first considered the Dirichlet-type problems for the integro-differential equations of fractional order.Classification of linear and non-linear partial differential equations of fractional order is still far from being completed. Several results for partial differential equations are described. Among these results, we mention the pioneering work by Gerasimov and recent books.Anyway,thisareaisrapidlygrowing,sincemostoftheresultsarerelatedtodifferenttypes of applications. It is impossible to describe all existing results. Partly, they are presented in. We also note that many authors have applied methods of fractional integro-differentiation to construct solutions to ordinary and partial differential equations, to investigate integro-differential equations and to obtain a unified theory of specialfunctions.

5. Mittag-Leffler Functions in Stochastic models:

Stochastic modeling, which uses the fractional calculus approach, as well as the machinery of the Mittag-Leffler functions, is connected mainly with the concept of the continuous time random walk(CTRW)CTRW was introduced by Montroll and Weiss as a generalization of physical diffusion process to effectively describe anomalous diffusion, i.e., the super- and sub-diffusive cases. An equivalentformulation of the CTRW is given by generalized master equations. A connection between CTRWs and diffusion equations with fractional time derivatives has been established. Similarly, time-space fractional diffusion equations can be considered as CTRWs with continuously distributed jumps or continuum approximations of CTRWs on lattices.

A fractional generalization of the Poisson probability distribution was presented by Pillai in his pioneering work. He introduced the probability distribution (which he called the Mittag-Leffler distribution) using the complete monotonicity of the Mittag-Leffler function. The concept of a geometrically infinitely-divisible distribution was introduced by Klebanov, Maniya and Melamed. Later, , Pillai introduced a discrete analogue of such a distribution (*i.e*., the discrete Mittag-Leffler distribution). Another possible variant of the generalizations of the Poisson distribution isthat introduced by Lamperti

The use of the CTRW as a stochastic process from which once could derivephysically consistent fractional-order ODEs and fractional-order PDEs was relianton the introduction of power-law tailed waiting time densities such as the Mittag-Leffler density. The use of exponential waiting time densities results in integer orderderivatives. The key difference between the exponential and Mittag-Leffler distributions isthat the former is memoryless. For the exponential distribution the `waiting time'
until the next jump is not dependent on how much time has already elapsed. Inthe case of the Mittag-Leffler distribution the longer that one has waited, the longerone expects to wait. This memory property, the memory of how much time one hasalready waited, becomes manifest through a fractional derivative in the governingevolution equation. The fractional derivative, like an integral, requires the knowledge
of the full history of the solution, but unlike an integral it expresses the time rateof change of this full history, and it applies different weighting to different parts ofthe history.
we derive the governing master equations of an SIR model from a stochastic process with general history-dependent infectivity and recovery.

We incorporate fractional derivatives into both the infective and recovery terms by choosing

φ(t) to be power-law distributed and ρ(t) related to our choice of φ(t). In particular, we take φ(t) to beMittag–Leffler distributed

$$\varphi(t) = \frac{t^{\alpha-1}}{\tau^{\alpha}} E_{\alpha,\alpha}(-\{\tfrac{t}{\tau}\}^{\alpha})$$

for 0 <α < 1, where $\tau$is a scaling parameter. This distribution has a power-law tail, such that

φt) $\sim t^{-\alpha-1}$for large values of t. Here, $E_{\alpha,\beta}(z)$ is the two-parameter Mittag–Leffler function, given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \mathrm{Re}\,(\alpha) > 0, \quad \beta \in \mathbb{C}$$

The corresponding function $\emptyset(t)$ is

$$\emptyset(t) = E_{\alpha,1}(-\{\tfrac{t}{\tau}\}^{\alpha})$$

Using the relation between the Riemann–Liouville fractional derivative and its inverse Laplace transform, we are able to express the first integral which yields the fractional-order

infectivity and recovery SIR model.

we have derived a fractional-order infectivity and recovery model using a stochastic process. The fractional derivatives arise as a consequence of taking an age of infection-dependent infectivity and recovery to be power-law-distributed. In doing so, we have shown how to incorporate fractional derivatives into the model without violating the physicality of the parameters of the model. Under appropriate limits, we are able to simplify this generalised fractional model to the fractional recovery and classic SIR models.

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