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Convex Relaxation of a Quadratic and Bilinear Matrix Inequalities

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Abstract: This is a convex relaxation of the mathematical theory of linear matrix inequalities and bilinear matrix inequalities, which are used in process control. It is demonstrated that many convex inequalities found in process control applications are LMIs. Proofs are provided to acquaint the reader with LMI and BMI mathematics. Applications of LMIs and BMIs in process control include control structure selection, robust controller analysis and design, and efficient experiment design. Review of LMI and BMI problem-solving software.

Keywords: Convex Relaxation, Quadratic, bilinear matrix

Introduction: It is a convex requirement to have a linear matrix inequality (LMI). As a result, optimization problems with convex objective functions and LMI constraints can be solved rather randomly using commercially available software. An LMI can take on a wide variety of forms. LMIs can be used to express a variety of constraints from control theory, including Lyapunov and Riccati inequalities, linear inequalities, convex quadratic inequalities, matrix norm inequalities, and more. Additionally, a single LMI with a bigger dimension can always be written as a set of numerous LMIs. LMIs can therefore be used to solve a wide range of optimization and control issues. The majority of interesting control problems that cannot be expressed in terms of an LMI can be expressed in terms of a more general type of inequality known as a bilinear matrix (BMI).

There are no pre-made methods for solving BMI problems since computations over BMI constraints are intrinsically more challenging than computations over LMI constraints. But for BMI issues, algorithms are being created, the best of which can be used for moderately complicated process control issues. LMIs and BMIs have been the prevailing paradigm for expressing optimization and control problems due to their numerous appealing theoretical aspects. Although LMI/BMIs are becoming more widely accepted in academia, they have had little impact on process control practise. One of the main reasons for this is that process control engineers are typically not familiar with the mathematics of LMI/BMIs, and there is no introductory text available to aid the control engineer in learning this mathematics.The research monograph by Boyd and colleagues is the sole text that, as of the time this study was written, covers LMIs in any depth.

This monograph provides a helpful guide for finding LMI results that are dispersed across the electrical engineering literature, but it is not a textbook for instructing process control engineers in the principles of LMIs. Furthermore, there is no text that specifically discusses BMIs. Quadratic and Bilinear Matrix Inequalities (BMI) arise in many areas, especially in control and combinatorial optimization, where they play a central role. In control, the BMI has been extensively studied as a general framework for many NP-hard problems. In combinatorial optimization, non-convex quadratic inequalities are essential because they can capture Boolean or integer constraints on variables. For example, the constraint $x, \in \{-1,1\}$ is equivalent to the two constraints $x_i^2 \leq 1, x_i^2, \geq 1$ where the second inequality is non-convex.

Consider the following problem with a quadratic matrix inequality:

$$\text{Minimize } c^T x$$

$$Subject\ to\ C + \sum_{i=1}^{k} x_i A_i + \sum_{i=1}^{k} x_i x_j B_{ij} \leq 0,$$

Where,$x \in R^n$ is the optimization variable, and $x \in R^n$ and the symmetric matrices $A_{i,} B_{ij\,,} x \in R^{n\times n}$ are given. This problem is very general, but also non convex. For example, if the matrices C C, A,and $B_{ij\,,}$ are diagonal, the constraint in reduces to a set of n (possibly indefinite) quadratic constraints in x Problem therefore includes all quadratic optimization problems. It also includes all polynomial problems (since by introducing new variables, one can reduce any polynomial inequality to a set of quadratic inequalities), and all {0,1} and integer programs.

In control theory, a more restricted bilinear form is often sufficiently general. Here we split the variables in two vectors and y, and replace the constraint by a bilinear matrix inequality (BMI):

$$\text{Minimize } c^T x + c^T y$$

$$subject\ to\ D + \sum_{i=1}^{m} x_i A_i + \sum_{k=1}^{I} y_k B_k \sum_{i=1}^{m} \sum_{k=1}^{I} x_i y_k C_{ik} \leq 0,$$

BMIs include a wide variety of control problems, including synthesis with structured uncertainty, and fixed-order and fixed-structure controller design.

We first express the problem as

$$\text{Minimize } c^T x$$

$$subject\ to\ C + \sum_{i=1}^{m} x_i A_i + \sum_{ij=1}^{m} w_{ij} B_{ij} \le 0,$$

$$w_{ij} = x_i x_j. \quad i,j = I, \ldots, m$$

The second constraint can be written as $W = xx^T$, this equality is equivalent to then following:

$$\text{Rank}\begin{bmatrix} W & x \\ x^T & I \end{bmatrix} = 1$$

since the block matrix above is rank one if and only if the Schur complement of the (2,2) block is equal to zero, $i.e., W - xx^T = 0$.To see this, we use the result that the rank of a Fermitian matrix is equal to the rank of a diagonal block plus the rank of its Schur complement, $i.e.$, if C is invertible, then

$$\text{Rank}\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \text{Rank } C + \text{Rank } (A - BC^{-1}B^T).$$

The constraint implies **Rank** 1+**Rank**$(W - xx^T) = 1$,or $W - xx^T$.

Therefore, problem is equivalent to

$$\text{Minimize } \quad c^T x$$

$$Subject\ to\ C + \sum_{i=1}^{m} x_i A_i + \sum_{ij=1}^{m} w_{ij} B_{ij} \le 0,$$

$$\text{Rank}\begin{bmatrix} W & x \\ x^T & I \end{bmatrix} \le 1$$

where, we have replaced the equality in above with an inequality, so that the problem has the required form.

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