ON THE APPLICATION OF MODIFIED LAGRANGEAN METHOD: MATHEMATICAL MODEL FOR MEASURING WELFARE

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Abstract

Welfare state of the people has been an elusive term to measure over the years and as such giving it a near quantitative measure has not been adequately addressed. The work, therefore, uses modified Lagrangean method to develop a theoretical welfare measurement model, which is used in measuring the welfare state of the people and setting up the right strategies for improving it. The results show that welfare state is dependent on the average society budgetary allocation and on the price of the products utilized. Any variation on these parameters brings about a significant change on the welfare state of the people. The work is, therefore, important to individuals in making decisions about their welfare state, to the government in setting up strategies for improving the welfare state of its people, and to the trade unions in their continuous tussle with employers in trying to improve the welfare state of the workers.

Key Words: welfare state, modified lagrangean method, welfare measurement, average budgetary allocation, price, welfare improvement.

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1. Introduction

Welfare is concerned with the evaluation of alternative economic situations from the point of view of the society's well being. For example, assume that the total current welfare in the society is W, and that with the available resources, welfare is improved to $\overset{*}{W}$, the task is to show how to achieve this improvement in welfare subject to resource constraints. Welfare is said to be improved when the greatest good is secured for the greatest number in the society. In this work, modified Lagrangean mathematical method has been used to develop a standard welfare measurement model that describes the welfare state of consumers in any society. The basic concepts used are budgetary constraint function and the utility function. The point where utility function is maximized subject to budgetary constraint for all individuals represents the equilibrium level of consumption for the society. The study achieves abstraction from reality using well set meaningful assumptions. In this study, it is important to note that many consumers' in the society pursue their individual economic goals while endeavoring to achieve their welfare state without regarding the needs of others. In this line, determining society's welfare position is not easy. It is because of this that an in depth study has been carried out to paint a clear picture of measuring welfare.

Various authors have studied the theory of Lagrangean methods, and where necessary have written papers on several applications of the same. Franklin [6] writes that one of the early works in this subject was developed by Karush in 1939. Karush derived the conditions for local optimality, which were later called the Kuhn-Tucker Conditions. In their work, Kuhn and Tucker [12] developed a fundamental theorem of mathematical programming giving necessary conditions for optimal points. Modified Lagrangean method was studied by Arrow and Solow [1]. In their study, the simplest quadratic modification on Lagrangean function was introduced for equality constrained non linear programming problem and continuous steepest descent, which was shown to converge locally to a saddle point under reasonable assumptions to the problem. In the same year, they also gave an economic interpretation of modified Lagrangean function. Later in 1961, Arrow and Enthoven as shown in Blaine and Schulze [3] gave a generalization of Kuhn and Tucker sufficient conditions for a maximum.

Inspired by the work done by Arrow and Solow [1], Hestenes [10] developed new iterative methods on modified Lagrangean function. The subject of his discussion though the same as Arrow and Solow [1], was an improvement on the same. Later, Wierzbicki [18] applied modified Lagrangean formulae to the solution of inequality constrained optimization problem. Polyak [16], separately, also applied modified Lagrangean formulae in solving linear and convex programming problems. A remarkable development on the work done by the two groups was by Rockafeller [17] who also applied modified Lagrangean formulae in studying both linear and convex programming. He found out that the merits of the method are due to its invaluable properties, which are more superior as compared to ordinary Lagrangean formulae. More applications of this method were also done on several known algorithms by Golshtein [8] and Mangasarian [15]. By this time, it had now been realized that the class of functions competitive with Lagrangean function are fairly wide. Influenced by the work of Golshtein and Tretyakov [9], two other mathematicians, Kort and Bertsekas [11] too described this class of functions. In their work, the notion of stability of saddle points was discussed in, which modified Lagrangean function for linear programming problem was constructed with the property of stability of saddle points. This was done with respect to both primal and dual variables.

Becker [4] used ordinary Lagrangean method to study household production approach to consumption. In this study, he argued that household receives utility from commodities but these commodities are not simply goods obtained from the market. The commodities consumed by households are 'produced' in the household by using purchased goods as well as the time of the household members. A related study was carried by McCallum [14]. He used modified Lagrangean in his discussion of monetary overlapping generations' model. He argued that generations overlap. Each generation lives two periods, youth and old age, and old age agents from the previous generation overlap with the youth of the current generation.

The literature on this subject has become scanty except for Golshtein and Tretyakov [7] where mainly Modified Lagrangeans and Monotone Maps in Optimisation are discussed. It is therefore worthwhile to develop a welfare measurement model, which can be instrumental in deriving welfare measurement consumption points of all members of the society, and describing the implications of variations of parameters involved in improving the welfare of the people.

2. Model Building and Assumptions

In order to realize the objectives of this study, this section brings into picture a simple mathematical model constructed with the aid of meaningful and consistent economic and mathematical assumptions. For example, the following assumptions guide us in developing and writing this model:

i. There are k members in the society with respective income endowments $Y_1, Y_2, \cdots, Y_k$.

ii. The levels of satisfaction of members are independent. There is no possibility of substituting the level of satisfaction of one member with that of another.

iii. The budgetary constraint is the only tangible constraint on the activities of the members.

iv. Social welfare is said to be maximized when the entire populace maximizes their individual utility subject to their budgetary allocations.

v. The utility level of each member is jointly proportional to the size of his basket of commodities.

vi. Consumption is such that each member of the society has the potential of affording at least one unit of commodities used in this particular society.

In this paragraph, the use of choice variables to represent the size of basket of commodities consumed by members of the community is introduced. We let the choice variables to be

$$x_1^j, x_2^j, \cdots, x_n^j, \quad j \in I = 1, 2, \cdots, k \qquad (2.1)$$

In the choice variables (2.1), x_n^j represent the quantity of the n^{th} commodity purchased by the j^{th} member of the society. Each of the units consumed by individuals is not free. They have to be bought at certain market prices. We let the price charged per unit of the content of a basket of commodities to be

$$p_i > 0, \quad i \in I = 1, 2, \cdots, n \qquad (2.2)$$

Considering variables (2.1) and price per unit (2.2), the amount paid by the j^{th} person to acquire the i^{th} commodity is given as

$$p_i x_i^j, \quad i \in I = 1, 2, \cdots n; \quad j \in I = 1, 2, \cdots, k; i \neq j \qquad (2.3)$$

Every consumer gets satisfaction after consuming some unit of a commodity. This satisfaction is called the consumer's utility. Therefore utility is expressed as a function of the choice variables.

Thus the utility function of the j^{th} person who consumes the basket of goods (2.1) is expressed as

$$U^j\left(x_1^j, x_2^j, \cdots, x_n^j\right), \quad j \in I = 1, 2, \cdots, k \qquad (2.4)$$

The consumer does not get the commodities free of charge. He has to pay for them. This means each consumer must have some budgetary allocation from which, he extracts liquid money and assign it to every unit of his basket of commodities in order of some preference. Suppose income in a particular community is distributed such that the j^{th} person's allocation is

$$Y^j, \quad j \in I = 1, 2, \cdots, k \qquad (2.5)$$

then such a consumer is expected to maximize his objective function (2.4) subject to the constraint function

$$p_1 x_1^j + p_2 x_2^j + \cdots + p_n x_n^j \le Y^j, \quad j \in I = 1, 2, \cdots, k \qquad (2.6)$$

The symbol $\le$ is used to mean the j^{th} consumer is expected to use not more than his budgetary allocation. We shall also need notations that distinguish between the choice variables (2.1) and some values of the optimum called optimal solutions. In this work, optimal solutions are represented by notations

$$\overset{*}{x}{}_1^j, \overset{*}{x}{}_2^j, \cdots, \overset{*}{x}{}_n^j, \quad j \in I = 1, 2, \cdots, k \qquad (2.7)$$

If we take into consideration the assumption that utility of each member is jointly proportional to the size of his basket of commodities, then the utility function of the j^{th} person is represented by the function

$$U^j = a_0 \left(x_1^j\right)^{a_1} \left(x_2^j\right)^{a_2}, \cdots, \left(x_n^j\right)^{a_n}, \quad j \in I = 1, 2, \cdots, k \qquad (2.8)$$

In the utility function (2.8), a_0 is a constant of proportionality where as $a_i > 0$, $i \in I = 1, 2, \cdots, n$ shows the proportion by which expenditure on the i^{th} commodity changes if the total income is changed by one unit.

$$\therefore \sum_{i=1}^{n} a_i = 1 \qquad (2.9)$$

It is also worth noting at this point that in this model a_i, $i \in I = 1, 2, \cdots, n$ are assumed to be the same for all individuals because of their common beliefs, cultural values and attitudes that impact on the consumption of commodities. If we now consider the utility function (2.4) and the constraint function (2.6), the optimization problem of the j^{th} person is formulated as

$$\left.\begin{array}{l} \max imize: U^j\left(x_1^j, x_2^j, \cdots, x_n^j\right) \\ x_1^j, x_2^j, \cdots, x_n^j \\ subject\,to: p_1 x_1^j + p_2 x_2^j + \cdots + p_n x_n^j \le Y^j \\ \qquad x_1^j \ge 0, x_2^j \ge 0, \cdots, x_n^j \ge 0; \quad j \in I = 1,2,\cdots,k \end{array}\right\} \quad (2.10)$$

In the formulation (2.10), an individual has the option of using not more than his budgetary allocation. Thus he has to make a decision to either use part of the income or use the whole of it. Also present in the formulation of the problem are the non-negativity constraints imposed on the choice variables. This problem is studied by the method of modified Lagrangean. If we decide to write the full Lagrangean without ignoring any constraint, then we get

$$L = U^j\left(x_1^j, x_2^j, \cdots, x_n^j\right) - \lambda^j\left[p_1 x_1^j + p_2 x_2^j + \cdots + p_n x_n^j - Y^j\right] + \sum_{k=1}^{n} \eta_k^j x_k^j, \quad (2.11)$$
$$j \in I = 1, 2, \cdots, k$$

The modified Lagrangean function $\overset{*}{L}$ is thus a function formed by ignoring all the non-negativity constraints of problem (2.10). Function (2.11) is therefore written as

$$\overset{*}{L} = U^j\left(x_1^j, x_2^j, \cdots, x_n^j\right) - \lambda^j\left[p_1 x_1^j + p_2 x_2^j + \cdots + p_n x_n^j - Y^j\right], \quad j \in I = 1, 2, \cdots, k \quad (2.12)$$

and in this case, the constraints $x_i^j \ge 0, \quad j \in I = 1, 2, \cdots, k; \; i \in I = 1, 2, \cdots, n \; and\, i \ne j$ have been ignored. Taking partial derivatives of function (2.12) gives the following systems of equations

$$\left.\begin{array}{l} \dfrac{\partial \overset{*}{L}}{\partial x_1^j} = \dfrac{\partial U^j}{\partial x_1^j} - \lambda^j p_1 \\ \dfrac{\partial \overset{*}{L}}{\partial x_2^j} = \dfrac{\partial U^j}{\partial x_2^j} - \lambda^j p_2 \\ \vdots \qquad \vdots \qquad \vdots \\ \dfrac{\partial \overset{*}{L}}{\partial x_n^j} = \dfrac{\partial U^j}{\partial x_n^j} - \lambda^j p_n, \quad j \in I = 1, 2, \cdots, k \end{array}\right\} \quad (2.13)$$

and

$$\frac{\partial \overset{*}{L}}{\partial \lambda^j} = Y^j - p_1 x_1^j - p_2 x_2^j - \cdots - p_n x_n^j, \quad j \in I = 1, 2, \cdots, k \qquad (2.14)$$

Since problem (2.10) satisfies the quasi concave programming conditions, then for it to have a unique solution the modified Lagrangean function (2.12) and its partial derivatives (2.13) to (2.14) must satisfy the Kuhn-Tucker conditions

$$\left.\begin{array}{l} if\ x_1^j \geq 0\ and\ \frac{\partial \overset{*}{L}}{\partial x_1^j} \leq 0\ then\ x_1^j \frac{\partial \overset{*}{L}}{\partial x_1^j} = 0 \\ \quad x_2^j \geq 0\ and\ \frac{\partial \overset{*}{L}}{\partial x_2^j} \leq 0\ then\ x_2^j \frac{\partial \overset{*}{L}}{\partial x_2^j} = 0 \\ \quad \vdots \qquad \vdots \qquad \vdots \\ \quad x_n^j \geq 0\ and\ \frac{\partial \overset{*}{L}}{\partial x_n^j} \leq 0\ then\ x_n^j \frac{\partial \overset{*}{L}}{\partial x_n^j} = 0, \quad j \in I = 1, 2, \cdots, k \end{array}\right\} \qquad (2.15)$$

and

$$if\ \lambda^j \geq 0\ and\ \frac{\partial \overset{*}{L}}{\partial \lambda^j} \leq 0\ then\ \lambda^j \frac{\partial \overset{*}{L}}{\partial \lambda^j} = 0, \quad j \in I = 1, 2, \cdots, k \qquad (2.16)$$

Suppose we let $x_1^j > 0, x_2^j > 0, \cdots, x_n^j > 0\ and\ \lambda^j > 0, \quad j \in I = 1, 2, \cdots, k$ then the conditions

$$\frac{\partial \overset{*}{L}}{\partial x_1^j} = \frac{\partial \overset{*}{L}}{\partial x_2^j} = \cdots = \frac{\partial \overset{*}{L}}{\partial x_n^j} = \frac{\partial \overset{*}{L}}{\partial \lambda^j} = 0, \quad j \in I = 1, 2, \cdots, k \qquad (2.17)$$

are satisfied and the system of equations (2.13) now becomes

$$\left.\begin{array}{l} \frac{\partial U^j}{\partial x_1^j} = \lambda^j p_1 \\ \frac{\partial U^j}{\partial x_2^j} = \lambda^j p_2 \\ \vdots \qquad \vdots \\ \frac{\partial U^j}{\partial x_n^j} = \lambda^j p_n, \quad j \in I = 1, 2, \cdots, k \end{array}\right\} \qquad (2.18)$$

The remaining condition $\frac{\partial \overset{*}{L}}{\partial \lambda^j} = 0,\ j \in I = 1, 2, \cdots, k$ of (2.17) implies that equation (2.14) becomes

$$p_1 x_1^j + p_2 x_2^j + \cdots + p_n x_n^j = Y^j, \quad j \in I = 1, 2, \cdots, k \qquad (2.19)$$

This implies that the constraint function is satisfied and the systems of equation (2.18) and the constraint function (2.19) are solved simultaneously to obtain optimal solutions. Suppose the objective function $U^j,\ j \in I = 1,2,\cdots,k$ is considered as defined in function (2.8) then the partial derivatives are found to be

$$\left.\begin{aligned} \frac{\partial U^j}{\partial x_1^j} &= a_1 \frac{U^j}{x_1^j} \\ \frac{\partial U^j}{\partial x_2^j} &= a_2 \frac{U^j}{x_2^j} \\ &\vdots \\ \frac{\partial U^j}{\partial x_n^j} &= a_n \frac{U^j}{x_n^j}, \quad j \in I = 1,2,\cdots,k \end{aligned}\right\} \qquad (2.20)$$

If we now compare the system of equations (2.18) and (2.20), we obtain equations (2.21), (2.22) and (2.23) as

$$a_1 U^j = \lambda^j p_1 x_1^j, \quad j \in I = 1,2,\cdots,k \qquad (2.21)$$

$$a_2 U^j = \lambda^j p_2 x_2^j, \quad j \in I = 1,2,\cdots,k \qquad (2.22)$$

and

$$a_n U^j = \lambda^j p_n x_n^j, \quad j \in I = 1,2,\cdots,k \qquad (2.23)$$

Adding equations (2.21) to (2.23) and substitute equation (2.19) it yields

$$\lambda^j = \frac{U^j}{Y^j}, \quad j \in I = 1,2,\cdots,k \qquad (2.24)$$

and if we substitute it in equations (2.21) to (2.23) the system of equations are obtained as

$$\left.\begin{aligned} p_1 x_1^j &= a_1 Y^j \\ p_2 x_2^j &= a_2 Y^j \\ &\vdots \\ p_n x_n^j &= a_n Y^j, \quad j \in I = 1,2,\cdots,k \end{aligned}\right\} \qquad (2.25)$$

Clearly, the system of equations (2.25) again shows that the j^{th} person spends the proportions $a_1, a_2, \cdots, a_n$ of his budgetary allocation on commodities $x_1, x_2, \cdots, x_n$ respectively regardless of the price. In view of the relation (2.25), the optimal solutions of the objective function (2.8) subject non-negativity and budgetary constraints functions described in problem (2.10) are

$$\left.\begin{aligned} \overset{*}{x}{}_1^j &= a_1 \frac{Y^j}{p_1} \\ \overset{*}{x}{}_2^j &= a_2 \frac{Y^j}{p_2} \\ \vdots \quad & \quad \vdots \\ \overset{*}{x}{}_n^j &= a_n \frac{Y^j}{p_n}, \quad j \in I = 1, 2, \cdots, k \end{aligned}\right\} \qquad (2.26)$$

The optimal solution (2.26) can, therefore, be written in a more general form as

$$\overset{*}{x}{}_i^{;j} = a_i \frac{Y^j}{p_i}, \quad j \in I = 1, 2, \cdots, k; \quad i \in I = 1, 2, \cdots, n; i \neq j \qquad (2.27)$$

and it shows the ratio of the proportion of an individual's budgetary allocation of a particular product to the price of that product. At this point, the consumer is said to be in equilibrium and enjoys maximum satisfaction within the boundaries of budgetary allocation. It is important to emphasize that since welfare state of an individual is judged based on his level of consumption, relation (2.27) is, therefore, a good measure of a person's welfare so long as the measurable parameters $a, Y \ and \ p$ are known.

At this juncture, we resort to recognizing the optimal consumption equilibrium point for the whole society. The society in this case is taken to comprise a group of people with the same attitudes, culture and beliefs that controls their socio economic operations. We argue that although members have different income levels, the products they use are the same both physically and intrinsically. What might be different could be the quantities of products they consume, which is brought about by difference in income. We thus adopt lateral summation in pursuit of the society's equilibrium consumption level. As such, we let $E_i, i = 1, 2, \cdots, n$ to be the society's equilibrium expenditure of the i^{th} commodity. From equations (2.25) it evident that the society's equilibrium levels of expenditure on each commodity is:

$$\left.\begin{aligned} E_1 &= p_1 x_1^1 + p_1 x_1^2 + \cdots + p_1 x_1^k \\ &= a_1 Y^1 + a_1 Y^2 + \cdots + a_1 Y^k \\ &= a_1 \left[Y^1 + Y^2 + \cdots + Y^k \right] \\ &= a_1 \sum_{j=1}^{k} Y^j \end{aligned}\right\} \qquad (2.28)$$

$$\left.\begin{aligned} E_2 &= p_2 x_2^1 + p_2 x_2^2 + \cdots + p_2 x_2^k \\ &= a_2 Y^1 + a_2 Y^2 + \cdots + a_2 Y^k \\ &= a_2 \left[Y^1 + Y^2 + \cdots + Y^k\right] \\ &= a_2 \sum_{j=1}^{k} Y^j \end{aligned}\right\} \qquad (2.29)$$

and

$$\left.\begin{aligned} E_n &= p_n x_n^1 + p_n x_n^2 + \cdots + p_n x_n^k \\ &= a_n Y^1 + a_n Y^2 + \cdots + a_n Y^k \\ &= a_n \left[Y^1 + Y^2 + \cdots + Y^k\right] \\ &= a_n \sum_{j=1}^{k} Y^j \end{aligned}\right\} \qquad (2.30)$$

By considering both solutions (2.28) to (2.30), we generate a general society equilibrium level of expenditure

$$E_i = a_i \sum_{j=1}^{k} Y^j, \quad i = 1, 2, \cdots, n \qquad (2.31)$$

and it shows that the society spends fixed proportions $a_1, a_2, \cdots, a_n$ of its total budgetary allocations on commodities $x_1, x_2, \cdots, x_n$ respectively regardless of the price charged.

We now need to similarly establish the society's equilibrium consumption level $\overset{*}{x}_i, \quad i = 1, 2, \cdots, n$ of the i^{th} commodity. From the solutions (2.26), it is evident that the society realizes its highest level of satisfaction when the equilibrium consumption levels of individual members are averaged. Thus, the point at which the society maximizes its welfare is computed as follows:

$$\left.\begin{aligned} \overset{*}{x}_1 &= \frac{\left[\overset{*}{x}{}_1^1 + \overset{*}{x}{}_1^2 + \cdots + \overset{*}{x}{}_1^k\right]}{k} \\ \overset{*}{x}_2 &= \frac{\left[\overset{*}{x}{}_2^1 + \overset{*}{x}{}_2^2 + \cdots + \overset{*}{x}{}_2^k\right]}{k} \\ &\vdots \qquad \vdots \\ \overset{*}{x}_n &= \frac{\left[\overset{*}{x}{}_n^1 + \overset{*}{x}{}_n^2 + \cdots + \overset{*}{x}{}_n^k\right]}{k} \end{aligned}\right\} \tag{2.32}$$

and this can be written in a more general form as

$$\overset{*}{x}_i = \frac{1}{k}\sum_{j=1}^{k} \overset{*}{x}{}_i^j, \quad i \in I = 1, 2, \cdots, n \tag{2.33}$$

The solution (2.33) can be made more robust by substituting in it $\overset{*}{x}{}_i^j, \quad i \in I = 1, 2, \cdots, n; \quad j \in I = 1, 2, \cdots k; i \neq j$ of solutions (2.26) so as to get

$$\left.\begin{aligned} \overset{*}{x}_1 &= \frac{1}{k}\left[\overset{*}{x}{}_1^1 + \overset{*}{x}{}_1^2 + \cdots + \overset{*}{x}{}_1^k\right] \\ &= \frac{1}{k}\left[a_1\frac{Y^1}{p_1} + a_1\frac{Y^2}{p_1} + \cdots + a_1\frac{Y^k}{p_1}\right] \\ &= \frac{a_1}{p_1}\left[\frac{Y^1 + Y^2 + \cdots + Y^k}{k}\right] \\ &= \frac{a_1}{p_1}\frac{1}{k}\sum_{j=1}^{k} Y^j \end{aligned}\right\} \tag{2.34}$$

$$\left.\begin{aligned} \overset{*}{x}_2 &= \frac{1}{k}\left[\overset{*}{x}{}_2^1 + \overset{*}{x}{}_2^2 + \cdots + \overset{*}{x}{}_2^k\right] \\ &= \frac{1}{k}\left[a_2\frac{Y^1}{p_2} + a_2\frac{Y^2}{p_2} + \cdots + a_2\frac{Y^k}{p_2}\right] \\ &= \frac{a_2}{p_2}\left[\frac{Y^1 + Y^2 + \cdots + Y^k}{k}\right] \\ &= \frac{a_2}{p_2}\frac{1}{k}\sum_{j=1}^{k} Y^j \end{aligned}\right\} \tag{2.35}$$

and

$$\left.\begin{aligned}\overset{*}{x}_n &= \frac{1}{k}\left[\overset{*}{x}_n^1 + \overset{*}{x}_n^2 + \cdots + \overset{*}{x}_n^k\right] \\ &= \frac{1}{k}\left[a_n\frac{Y^1}{p_n} + a_n\frac{Y^2}{p_n} + \cdots + a_n\frac{Y^k}{p_n}\right] \\ &= \frac{a_n}{p_n}\left[\frac{Y^1 + Y^2 + \cdots + Y^k}{k}\right] \\ &= \frac{a_n}{p_n}\frac{1}{k}\sum_{j=1}^{k} Y^j\end{aligned}\right\} \qquad (2.36)$$

If we consider the solutions (2.34) to (2.36), then in general the society maximizes its welfare when the equilibrium consumption level of the i^{th} commodity is given as

$$\overset{*}{x}_i = \frac{a_i}{p_i}\frac{1}{k}\sum_{j=1}^{k} Y^j, \quad i \in I = 1, 2, \cdots, n \qquad (2.37)$$

This equilibrium consumption level is a welfare measurement quantity for the society. It shows that its value depends on the average societal budgetary allocation to a particular product and the price of that particular product.

There are situations when certain constrained functions yield solutions at the corners of the bounded region. We therefore use the Kuhn-Tucker Conditions (2.15) and (2.16) to investigate the possibility of the modified Lagrangean function (2.12) accepting other solutions. Suppose the modified Lagrangean function is written as

$$\overset{*}{L} = a_0 \prod_{i=1}^{n}\left(x_i^j\right)^{a_i} + \lambda^j\left[Y^j - \sum_{i=1}^{n} p_i x_i^j\right], \quad j \in I = 1, 2, \cdots, k \qquad (2.38)$$

and if we take it partial derivatives with respect to $x_i^j, \quad i \in I = 1, 2, \cdots, n; \quad j \in I = 1, 2, \cdots, k; i \neq j$ and $\lambda^j, \quad j \in I = 1, 2, \cdots, k$ we obtain

$$\frac{\partial \overset{*}{L}}{\partial x_i^j} = a_i\frac{U^j}{x_i^j} - \lambda^j p_i \qquad (2.39)$$

and

$$\frac{\partial \overset{*}{L}}{\partial \lambda^j} = Y^j - \sum_{i=1}^{n} p_i x_i^j \qquad (2.40)$$

If we let $x_i^j = 0, i \in I = 1,2,\cdots,n;\ j \in I = 1,2,\cdots,k; i \neq j$ and $\lambda^j > 0\ \forall\ j$, then using the Kuhn-Tucker conditions, equations (2.39) and (2.40) becomes

$$a_i \frac{U^j}{x_i^j} - \lambda^j p_i < 0 \tag{2.41}$$

and

$$\sum_{i=1}^{n} p_i x_i^j = Y^j \tag{2.42}$$

respectively. Considering inequality (2.41), since $x_i^j = 0,\ i \in I = 1,2,\cdots,n;\quad j \in I = 1,2,\cdots,k$ then $a_i U^j < 0$; but $a_i > 0\ \forall\ i$, implying that $U^j < 0\ \forall\ j$. We have therefore reached a contradiction because the utility function of any member of the society cannot be negative. Therefore, our maximization problem does not accept corner solutions.

3. Results, Discussions and Interpretations

In this work, welfare state under non-equality of constraint analysis has been investigated. The analysis has used modified Lagrangean method and has justified its suitability based on how best it satisfies the Kuhn-Tucker conditions. Fortunately, the optimal consumption point for each individual given a basket of goods is found as

$$\overset{*}{x_i^j} = \frac{a_i Y^j}{p_i}, \quad i \in I = 1,2,\cdots,n; \quad j \in I = 1,2,\cdots,k; i \neq j \tag{3.1}$$

The solution (3.1) shows that welfare state of an individual depends on the proportion of his budgetary allocation assigned to a particular product, and on the price of that product. At this level of consumption, the consumer is said to be in equilibrium and enjoys maximum satisfaction within the boundaries of his budgetary allocation. This solution is discussed in six scenarios. First, the proportion of budgetary allocation is kept constant while the price is rising. Second, the proportion of budgetary allocation is rising while the price is kept constant. Third, proportion of budgetary allocation and the price are both increasing by the same proportion. Fourth, proportion of budgetary allocation and the price are both decreasing by the same proportion. Fifth, proportion of budgetary allocation and the price are both increasing by different proportions. Sixth, proportion of budgetary allocation and the price are both declining by different proportions. In the first scenario, if the proportion of budgetary allocation is kept constant while the price is rising, the size of the individual basket of goods will be contracted, meaning that the welfare state will go falling. In the second scenario, if the proportion of budgetary allocation is rising while the price is kept constant, the size of an individual basket of goods will go to rise, meaning that the welfare state will be improved. In the third scenario, if the proportion of budgetary allocation and price are both rising by same proportion, the size of the basket of goods will not change, meaning that the welfare state will be stable. In the fourth scenario, if the proportion of budgetary allocation and the price are both declining by the same proportion, the size of the basket of goods will not change, meaning that the welfare state will be stable. In the fifth scenario, if the budgetary allocation and the price are both increasing at different proportions, two situations are likely to a rise. One, the budgetary allocation may be rising by a bigger proportion than the price. Two, the budgetary allocation may be rising by a smaller proportion than the price. In situation one, the size of the basket of goods will go on rising, meaning that the welfare state will be improved. In situation two, the size of the basket of goods will go on falling, meaning that the welfare state will be declining. In the sixth scenario, if the budgetary allocation and the price go on declining at different proportions, two situations are again likely to a rise. One, the budgetary allocation may be falling by a smaller proportion than the price. Two, the budgetary allocation may be falling by a larger proportion than the price. In the first situation, the size of the basket of goods will not change or will increase depending on how small the proportion by which the budgetary allocation is falling in relation to a fall in price, meaning that the welfare state will either be maintained or improved. In situation two, the size of the basket of goods will fall making the welfare state to decline.

The work has also investigated the welfare state of all members of the society subject to their budgetary allocation. The investigation started by first establishing expenditure level on a particular product and the results was

$$E_i = a_i \sum_{j=1}^{k} Y^j, \quad i \in I = 1, 2, \cdots, n \qquad (3.2)$$

From results (3.2), it shows that the society needs the proportion $a_i, \quad i \in I = 1, 2, \cdots, n$ of its budgetary allocation to consume the i^{th} commodity. The study further found that the society's consumption level of a particular product is

$$\overset{*}{x}_i = \frac{a_i}{p_i}\frac{1}{k}\sum_{j=1}^{k} Y^j, \quad i \in I = 1, 2, \cdots, n \qquad (3.3)$$

The quantity represented by the results (3.3) depends on the price of a particular product and the proportion of the average society budgetary allocation to that product. This result is discussed in six scenarios. First, proportion of the average society budgetary allocation is kept constant while price increases. Second, proportion of the average society budgetary allocation is increasing while price is kept constant. Third, the proportion of the average society budgetary allocation and the price are both increasing by the same proportion. Fourth, the proportion of the average society budgetary allocation and the price are both decreasing by the same proportion. Fifth, the proportion of the average society budgetary allocation and the price are both increasing by different proportions. Sixth, the proportion of the average society budgetary allocation and price are both decreasing by different proportions. In the first scenario, if the proportion of the average society budgetary allocation is kept constant while the price is increasing, this will cause the size of the basket of goods to contract with the overall effect of lowering the welfare state. In the second scenario, if the proportion of the average society budgetary allocation is increasing while the price is kept constant, this will make the size of the basket of goods to expand with the overall effect of raising the welfare state. In the third scenario, if the proportion of the average society budgetary allocation and the price are both increasing by the same proportion, there will be no extension in the size of basket of goods, meaning that the welfare state will remain stagnant. In the fourth scenario, if the proportion of the average society budgetary allocation and the price are both decreasing by the same proportion, there will be no contraction in the size of the basket of goods, meaning that the welfare state will remain stable. In the fifth scenario, if the proportion of the average society budgetary allocation and the price are both increasing by different proportions, there are two reasoning in such a case. If the proportion by which the average society budgetary allocation is rising is faster than the proportion by which the price rising, the size of the basket of goods will be extended, meaning that the welfare state will improve. If on the other hand the proportion by which the proportion of the average society budgetary allocation is rising is slower than the proportion by which price is rising, the size of the basket of goods will contract, meaning the welfare state will have to decline. In the sixth scenario, if the proportion of the average society budgetary allocation and the price are both decreasing by different proportions, there are two reasoning in such a case. If the proportion of the average society budgetary allocation is falling by a proportion less than the proportion by which the price is falling, the size of the basket of goods will not contract, meaning that the welfare state will either be maintained or improved depending on the size by which the average society budgetary allocation is falling in relation to a fall in price. If on the other hand the proportion of the average society budgetary allocation is falling by the proportion more than the proportion by which the price is falling, the size of the basket of goods will contract, meaning that the welfare state is likely to fall.

The study also sought to find out if there could be other solutions different from the ones stated above. To investigate this, Kuhn-Tucker conditions as used in this work are appropriately applied. In this case, at all levels, each of the choice variables was set equal to zero at a time, while the others are strictly set positive. In all cases, the utility function of every member was found to be negative, which in essence is a contradiction. The level of satisfaction of any person cannot be negative. This rules out the possibility of having any other solution, in particular the corner solutions.

4. Conclusion

In this paper, a welfare measurement model has been developed and studied. The results show that the welfare state of the people is dependent on the average society budgetary allocation and the prices of commodities. Any variation on either average budgetary allocation or price brings a significant change on the welfare state of the people.

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