APPLICATION OF LAPLACE TRANSFORMS ON FIRST ORDER EQUATIONS: MATHEMATICAL MODEL FOR DEVELOPING AND ANALYZING STABILITY OF THE WAGE FUNCTION

Olala, Gilbert Owuor*

Abstract:

In this paper a first order wage equation is developed and solved by Laplace Transforms. The subsequent wage function is then analyzed and interpreted for stability. The function could initially stand off the equilibrium wage rate but in the long run, it asymptotically stabilizes in inter temporal sense. At times, the free market forces are known to cause uncertainties when volatility in wage rate is experienced and this may affect both investments and employment if not controlled. We therefore propose creating a middle path in which wage rate is allowed to oscillate freely within a narrow band managed by employers in consultation with the workers under the watch of the government.

Key words: wage equation, wage function, wage rate, equilibrium wage rate, stability, market forces, volatile wage rate, and middle path.

* Department of Mathematics & Computer Science, Kisumu Polytechnic, P. O. Box 143, 40100, Kisumu, KENYA

1. Introduction

Wages mean the reward for labor services. Wages may be categorized according to the mode of payment, status of the laborer, size of the wage, in nominal sense or in real sense. In terms of payment, wages can be paid hourly, daily, monthly or per unit of the produce. According to the status of the laborer, wage takes the form of pay, salary, fee or wage. Pay is the form of wage paid to highly qualified persons, for example, governors and commissioners. Salary is a form of wage paid to semi-qualified persons, for example accountants and secretaries. Professionals such as doctors, lawyers and doctors are paid fees while unskilled laborers working in factories earn wage. In terms of wage size, wages are termed as relative wages. This is evident where laborers of the same profession or different professions earn wages, which are not identical. In nominal and real sense, monetary wages are called nominal wages, while the total amount of necessities, comforts and other facilities, which a laborer may enjoy during his/her working life is called real wages. The real wage of a worker depends on the purchasing power of money, extra earnings of a worker on regular employment, conditions of work and expenses of the profession. In fact, any given time the purchasing power of money is stronger, real wage will automatically go up. On the other hand, if the purchasing power of money is weak, then real wage will go down. If any laborer can earn any extra income above his nominal wage, then his/her real wages will go up. On the other hand, if a laborer earns nothing above his/her nominal wage, then his/her real wages will be relatively smaller. Laborers who have regular employment are likely to earn higher real wages than those with irregular employment however large their nominal wage may be. It is therefore important that a laborer should earn regular wage however small for his/her to eventually get higher real wages in the long run. Laborers who work under reasonable conditions tend to earn higher real wage than those who work in severe conditions. Those who work in severe conditions tend to incur a lot of expenses on medical services and thus reduce the size of their nominal wage. In cases where expenses incurred running a given profession are higher, real wages tend to go down.

In this paper, we consider modeling a wage equation into a first order ordinary differential equation. For example, we consider the deterministic price adjustment model in [6]. In this study, a fixed supply and demand functions at instantaneous price for security are discussed. It is argued that at equilibrium asset price, the quantity demanded is equal to the quantity supplied.

This is discussed using the assumptions of fixed demand and supply curves while the price is kept constant. It also asserts that away from the equilibrium, excess demand for security raises its price, and excess supply lowers its price. In this situation, it is argued that the sign for the rate of change of price with respect to time depends on the sign of excess demand. If the demand and supply functions are made linear about a constant equilibrium price, the deterministic model of price adjustment is realized with respective sensitivities. This model is also called the deterministic logistic first order differential equation in price. In their analysis of the solution of the deterministic logistic equation, it was observed that in the long run the asset price settles at a constant steady state point at which no further change can occur.

In [4] a natural decay equation is developed. The equation describes a phenomenon where a quantity gradually decreases to zero. In the work, it is emphasized that convergence depends the sign of the proportionate parameter. If the proportionate parameter is negative then it turns into a growth equation and if it is positive it stabilizes in the long run. In the study of slope fields for autonomous equations the qualitative properties of the decay equation is demonstrated. It was found that the solution could be positive, negative or zero. In all the three cases, the solution approaches zero in limit as time approaches infinity. In the same work, models representing Newton's law of cooling, depreciation, population dynamics of diseases and water drainage are also presented with similar property in the long run.

In [1], the dynamics of market prices are studied. It was found out that if the initial price of the price function lies off the equilibrium point, then in the long run the price stability will be realized at the equilibrium position. In [2], equilibrium solutions representing a special class of static solutions are discussed. The study found that if a system starts exactly at equilibrium condition, then it will remain there forever. The study further found that in real systems, small disturbances often a rise which moves a system away from the equilibrium state. Such disturbances, regardless of their origin give rise to initial conditions which do not coincide with the equilibrium condition. If the system is not at equilibrium point, then some of its derivatives will be non zero and the system therefore exhibits a dynamic behavior, which can be monitored by watching orbits involved in the phase space.

The resistance-inductance electric circuit for constant electromotive force is modeled into a differential equation in [3].The stability of the solution is studied in the long run and is found to be a constant, which is the ratio of the constant electromotive force to the resistance.

The literature is silent and as a result it is worthwhile to develop a first order ordinary wage equation, solve it and establish its stability. The solutions of first order linear ordinary differential equations using Laplace transforms are presented in [5; 7] and this paper therefore uses the same in executing the solution of the wage equation.

2. Modeling the first order wage equation

In this section, we consider modeling a first order differential wage equation. The number of laborers demanded N_d is given by the linear function

$$N_d = \eta - \sigma W, \qquad (\eta, \sigma > 0) \qquad (2.1)$$

where η is a parameter showing the number of laborers demanded that does not depend on the wage rate W and σ is a parameter that shows the proportion by which the number of laborers demanded N_d responds to variation in wage rate. The number of laborers supplied N_s is given by

$$N_s = -\theta + \lambda W, \qquad (\theta, \lambda > 0) \qquad (2.2)$$

where θ is a parameter showing the number of laborers supplied that does not depend on the wage rate W and λ is a parameter that shows the proportion by which the number of laborers supplied N_s responds to variation in wage rate. At equilibrium, the number of laborers demanded must equal the number of laborers supplied; that is

$$N_s = N_d \Rightarrow -\theta + \lambda W = \eta - \sigma W$$

$$\Rightarrow W(\lambda + \sigma) = \eta + \theta$$

$$\therefore \hat{W} = \frac{\eta + \theta}{\lambda + \sigma}; \quad \lambda \neq -\sigma$$

(2.3)

The solution (2.3) is called an equilibrium wage rate of a laborer. If it happens that the initial wage rate is $\hat{W}$, then labor market is said to be clearly at equilibrium. At this point no dynamical analysis will be necessary.

In a more interesting case, the initial wage rate may not be equal to $\hat{W}$. The value $\hat{W}$ is therefore attainable after some due process of adjustments. During this time, not only will the wage rate change, but since N_d, the number of laborers demanded and N_s, the number of laborers supplied also depends on the wage rate, they must also change. Under such circumstances, the wage rate and quantity variables are considered to be functions of time.

Let's suppose that the wage rate is given sufficient time for adjustment, then we can investigate its long term trend. This therefore calls for finding the time path of the wage rate, which is governed by the relative strength of the forces of demand and supply of labor. Suppose the rate at which wage rate changes at any moment is directly proportional to excess demand prevailing at the moment, then mathematically this can be written as

$$\frac{dW}{dt} = \xi (N_d - N_s) \quad \xi > 0 \tag{2.4}$$

The parameter ξ in equation (2.4) represents an adjustment constant of proportionality. This equation can be such that $\frac{dW}{dt} = 0$ if the number of laborers demanded is equal the number of laborers supplied. In this scenario, the equilibrium wage rate may be looked at in inter temporal sense and in the market clearing sense. In inter temporal sense; the wage rate is viewed as constant over a long period of time. In the market clearing sense, the equilibrium wage rate is viewed when the number of laborers demanded equals the number of laborers supplied.

If we consider the demand function (2.1) and the supply function (2.2), equation (2.4) can be expressed as

$$\left.\begin{aligned}\frac{dW}{dt} &= \xi\left[(\eta-\sigma W)-(-\theta+\lambda W)\right]\\ &= \xi\left[(\eta+\theta)-(\lambda+\sigma)W\right]\\ &= \xi(\eta+\theta)-\xi(\lambda+\sigma)W\end{aligned}\right\} \tag{2.5}$$

Equation (2.5) can easily be written as a first order linear differential equation

$$\frac{dW}{dt}+\xi(\lambda+\sigma)W=\xi(\eta+\theta) \tag{2.6}$$

and this is the required wage equation.

3. Solution of first order wage equation

In this sub section, we write equation (2.6) as an initial value problem; that is,

$$\frac{dW}{dt}+\xi(\lambda+\sigma)W=\xi(\eta+\theta),\quad W(t)\big|_{t=0}=W_0 \tag{3.1}$$

Let $a = \xi(\sigma+\lambda)$ and $b = \xi(\eta+\theta)$ so that we have a simplified version of (3.1) written as

$$\frac{dW}{dt}+aW=b,\quad W(t)\big|_{t=0}=W_0 \tag{3.2}$$

This is solved by taking the Laplace transforms of the equation, that is,

$$\left.\begin{aligned}L\left(\frac{dW}{dt}\right)&=s\overline{W}-W_0\\ L(W)&=\overline{W}\\ L(b)&=\frac{b}{s}\end{aligned}\right\} \tag{3.3}$$

so that the differential equation (3.2) after inserting the initial conditions becomes

$$s\overline{W}-W_0+a\overline{W}=\frac{b}{s}$$

$$\Rightarrow (s+a)\overline{W}=W_0+\frac{b}{s}$$

$$\therefore\quad \overline{W}=\frac{sW_0+b}{s(s+a)}\qquad s\neq 0,-a \tag{3.4}$$

The right hand side of equation (3.4) must be separated into partial fractions so that

$$\overline{W} = \frac{b}{a}\left(\frac{1}{s}\right) + \left(W_0 - \frac{b}{a}\right)\left(\frac{1}{s+a}\right) \qquad (3.5)$$

We then take the inverse Laplace transforms of equation (3.5) to obtain $W(t)$, i.e.

$$\left.\begin{aligned} W(t) &= \frac{b}{a}L^{-1}\left(\frac{1}{s}\right) + \left(W_0 - \frac{b}{a}\right)L^{-1}\left(\frac{1}{s+a}\right) \\ &= \frac{b}{a} + \left(W_0 - \frac{b}{a}\right)\exp(-at) \end{aligned}\right\} \qquad (3.6)$$

But a $= \xi(\sigma + \lambda)$ *and b* $= \xi(\eta + \theta)$. This implies that

$$W(t) = \left(\frac{\eta+\theta}{\sigma+\lambda}\right) + \left(W_0 - \frac{\eta+\theta}{\sigma+\lambda}\right)\exp[-\xi(\sigma+\lambda)t] \qquad (3.7)$$

If we consider the equilibrium wage rate $\hat{W} = \frac{\eta+\theta}{\sigma+\lambda}, \sigma \neq -\lambda$, then solution (3.7) can be written in terms of $\hat{W}$ as

$$W(t) = \hat{W} + \left(W_0 - \hat{W}\right)\exp[-\xi(\sigma+\lambda)t] \qquad (3.8)$$

which is the required solution.

4. Results, analysis and interpretation

In this section, the results of this study are presented, analyzed and interpreted. In this paper an equilibrium wage rate

$$\hat{W} = \frac{\eta+\theta}{\sigma+\lambda}, \sigma \neq -\lambda \qquad (4.1)$$

has been established. In this case, θ is a parameter that shows the number of laborers supplied that does not depend on the wage rate, λ is a parameter that shows the proportion by which the number of laborers supplied responds to variation in wage rate, η is a parameter that shows the number of laborers demanded that does not depend on the wage rate and σ is a parameter that shows the proportion by which the number of laborers demanded responds to variation in wage rate. The wage equation

$$\frac{dW}{dt} + \xi(\lambda + \sigma)W = \xi(\eta + \theta) \qquad (4.2)$$

has also been established and it shows the equilibrium $\hat{W}$ may also be obtained after some process of adjustments. During this time, it is not only the wage rate which is expected to change but also the number of laborers demanded and supplied. It is important to note that the number of laborers supplied and demanded depends on the wage rate, which is a function of time. To obtain equation (4.2), the wage rate is assumed to change at any moment and is directly proportional to the excess demand. In this paper, the equation (4.2) with the initial condition $W(t)|_{t=0} = W_0$ has been fortunately solved by Laplace transforms to obtain

$$W(t) = \hat{W} + \left(W_0 - \hat{W}\right)\exp\left(-\xi(\sigma+\lambda)t\right), where\, \hat{W} = \frac{\eta+\theta}{\sigma+\lambda}, \sigma \neq -\lambda \quad (4.3)$$

We now investigate function (4.3) for stability by finding its path in the long run, i.e.

$$W(t) = \lim_{t\to\infty}\left[\hat{W} + \left(W_0 - \hat{W}\right)\exp\left(-\xi(\sigma+\lambda)t\right)\right] \quad (4.4)$$

In this case, $\left(W_0 - \hat{W}\right)$ is a constant and the value of the limit function (4.4) depends on the key factor $\exp\left(-\xi(\sigma+\lambda)t\right)$. In view of the fact that $\xi(\sigma+\lambda) > 0$ $\left(W_0 - \hat{W}\right)\exp\left(-\xi(\sigma+\lambda)t\right) \to \infty\ as\ t \to \infty$. The limit function (4.4) therefore becomes

$$W(t) = \hat{W}, \quad where\, \hat{W} = \frac{\eta+\theta}{\sigma+\lambda}, \sigma \neq -\lambda \quad (4.5)$$

This means that the time path of the wage function (4.3) moves towards the equilibrium position in the long run. This is interpreted in an inter-temporal sense rather than the market clearing sense. Further analysis of function (4.3) is possible by considering the relative positions of W_0 and $\hat{W}$; that is, comparing the relative positions of the initial wage rate and the equilibrium wage rate. This may be discussed in three different cases.

CASE I: In this case, we let $W_0 = \hat{W}$. This means that the wage function (4.3) becomes $W(t) = \hat{W}$. The time path of the function is thus constant and parallel to the time axis. The wage function is therefore in a stable state.

CASE II: In this case, we let $W_0 > \hat{W}$. The second term on the right hand side of function (4.3) is positive but it decreases as $t \to \infty$ since it is lowered by the value of the exponential factor $\exp\left(-\xi(\sigma+\lambda)t\right)$. The time path thus asymptotically approaches the equilibrium value $\hat{W}$ from above and in the long run, the wage function becomes stable.

CASE III: In this case, we let $W_0 < \hat{W}$ i.e. the initial wage rate is taken to be less than the equilibrium wage rate. The second term on the right hand side of function (4.3) is negative and it infinitely makes W_0 to rise asymptotically towards the equilibrium wage $\hat{W}$ *as* $t \to \infty$. These three cases are illustrated as shown in figure 4.1.

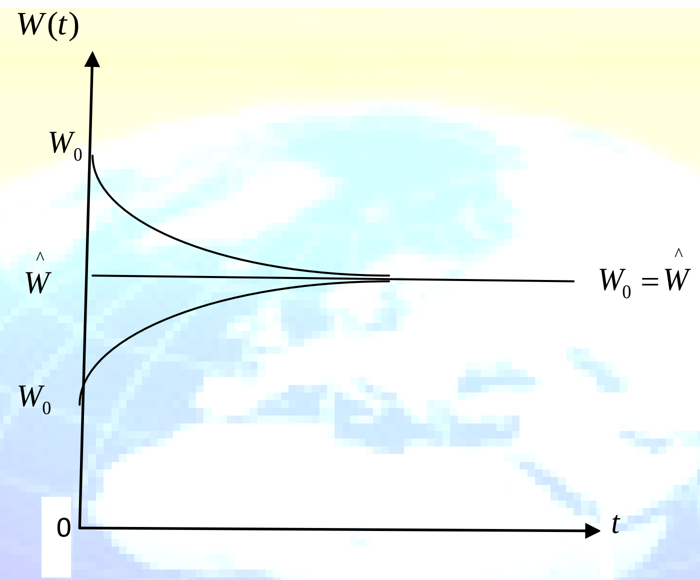

Figure 4.1: Stability Analysis of the Wage Function

Figure 4.1 shows that when $W_0 = \hat{W}$, *then* $W(t) = \hat{W}$, which is a constant function. If $W_0 > \hat{W}$, *then* $W(t)$ decreases asymptotically towards $\hat{W}$ and if $W_0 < \hat{W}$, *then* $W(t)$ increases asymptotically towards the equilibrium wage rate $\hat{W}$. The results therefore show that as $t \to \infty$, the function (4.3) approaches the equilibrium wage rate asymptotically either from above or below and stabilizes at the equilibrium wage rate.

5. Conclusion

In this paper, a first order wage functions has been developed and solved for the first time. The subsequent wage function is then analyzed and interpreted for stability. It is found that the wage function could stand off the equilibrium wage rate initially and as time approaches infinity, it asymptotically stabilizes at the equilibrium wage rate in inter temporal sense.

It has also been realized that where the wage rate is determined by the market forces of demand and supply, volatility in wage rate may be observed if it is not controlled. This may increase uncertainties and cause anxiety about investments and employment in the economy. The paper therefore proposes government intervention in the free operations of market forces in order to regulate the volatility of the wage rate. To do this, contrary to flexible wage rates influenced by forces of demand and supply, the paper proposes a middle path in which the wage rate is only allowed to operate within a narrow band managed by the employers in consultation with the workers under the government watch. According to this model, it suggests that whenever the wage rate hits at any of the boundaries of the band, employers must consult with the workers to keep it within the band. Therefore, movements of the wage rate within the narrow band will thus reflect the operations of the market forces and at the same time giving wage rate stability.

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