MODELING OF MAXIMUM FLOW AND SHORTEST PATH PROBLEMS IN DYNAMIC NETWORKS

Sahar Abbasi*

Fahimeh Moosavi*

Abstract.

This paper considers dynamic network in which, the flow commodity is dynamically generated at a source node and dynamically consumed at a sink node and the arc-flow bounds are time dependent. Then, the maximum dynamic flow problem in such networks for a pre-specified time horizon T is defined and mathematically formulated in both arc flow and path flow presentations. By exploiting the special structure of the problem, is showed to solve the general form of the dynamic problem as a minimum cost static flow problem and the generic augmenting path algorithm is used for solving this problem.

Keywords: dynamic networks, maximum flow, minimum cost network flow

* Department of Industrial Engineering,Najafabad Branch, Islamic Azad University,Isfahan, Iran.

1. Introduction

The importance and wide applicability of network flow problems has long been recognized. Dynamic network flow problems, in particular, have been used to model numerous real world phenomena arising in applications within almost all industries. Such applications include, for example, production-distribution systems, fleet management, road or air traffic control, production systems and communication [1]. In a *dynamic* network, the characteristics parameters, such as arcs capacity, flow cost and nodes supply/demand are constant values independent of time. However, dynamic network flow problems have been traditionally considered in a purely static environment, i.e. the attributes of the network, including arc traversal times (or costs), arc and node capacities, and the availability of supply, are time-invariant. For many of the aforementioned applications, such a static representation may be inadequate. Thus, to more realistically model such problems, the methodology must recognize the inherent time-varying nature not only of flow, but also of the network attributes. The dynamic shortest path problem is a generalization of the shortest path problem whose aim is to find a path of minimum cost (length) through a network for which

– Each arc has a transit time which specifies the amount of time to traverse through each arc,

– Parking (or waiting) is permitted at the nodes of the network for later departure,

and

– Network characteristics such as arc transit times and costs (or length) can change over time and are known for all values of time.

The general properties and algorithms have been discussed in both discrete time and continuous time settings by Ahuja et al. [2], Cai et al. [3], Chabini [4] Orda and Rom [5,6], Pallottino and Scutella [7], Philpott [8], and Philpott and Mees [9,10] among others. Dynamic flows are widely used in modeling of control processes from different technical, economic and informational systems. This class of network flows was first introduced by Ford and Fulkerson [14]. Examples and further applications can be found in the literature such as, Aronson [1], Powell et al. [11], Hoppe [12] and Fonoberova [13].This paper consists of modeling and analyzing the problem of maximum dynamic flow and dynamic shortest path on a dynamic network consisting of a source node s, a sink node d and a pre-specified integral time horizon T. We consider the network flows in the general form with time-varying arc capacities and bounds on arcs flow. In the network flows, the amount of flow on an arc changes at every moment during a pre-specified time horizon T. In this paper, we have used the generic augmenting path algorithm for studying the above problem in dynamic network flows. As it will be showed, the maximum flow problem on dynamic networks, in general case, can be formulated as a linear programming model whose special structure lead to an efficient solving algorithm.

In Section 2, we first give a formal definition of the dynamic maximum flow and shortest path problem. In Section 3, the proposed problem is formulated in two cases; continuous and discrete time. In Section 4, we describe a new time expanded network representation of a dynamic network flow which can be considered as a static network flow. Moreover, explain how to solve the dynamic network flow problem on this static network with the generic augmenting path algorithm. We summarize our conclusions and discuss the related problems for future research in Section 5.

2. Basic definitions and notation

In this section, we provide the basic concepts describing dynamic network flows with a source node **s** and a sink node **d**. Assume that G= (V, E, T) is a network flow with node setV, arc set E and an integral time horizon T. There are time-varying arc capacities $u_{ij}(t)$ which is interpreted as an upper bound on the rate of flow entering arc(i, j) ϵ E at time t, horizon capacities $\mathbf{u}_{ij}$, time-varying lower bounds $l_{ij}(t)$ assigned to each arc (i, j) ϵ E. and a non-negative transit time function λ_{ij} which determines the time it takes for flow to traverse arc(i, j). Finally, the horizon T is the time until which the flow can travel in the network. An dynamic s-d-flow with time horizon T is given by functions f_{ij}:[0,T)→R^+ ,for(i, j) ϵ E where $f_{ij}(\sigma)$ defines the rate of flow (per unit time) entering arc (i, j) at time σ. This flow arrives at the head node of (i, j) at time $\sigma + \lambda_{ij}(f_{ij}(\sigma))$.[14]

3. The maximum dynamic flow problem

We can formulate the dynamic flow problem in two ways depending on whether we use a discrete or continuous representation of time. In a continuous-time dynamic flow problem we look for the flow which distributed continuously over time within the time horizon T.The discrete-time dynamic flow problem is a discrete-time expansion of a static network flow problem. In this case we distribute the flow over a set of predetermined time steps t = 0, 1, . . . , T – 1.

3.1. The discrete-time model

A dynamic network flow G = (V, E, T) with discrete-time consists of a set of nodes V, (|V| = n), a set of arcs E, (|E| = m), two capacity functions u: E× N→R^+ and **u**: E→R^+ where N = {0, 1,. . . , T-1} is a set of time steps. In this case if x(t):E→R^+ satisfies the following constraints then it is feasible flow.

$$\sum_{j}\sum_{\sigma_1}^{\sigma_2} x_{ij}(t) - \sum_{j}\sum_{\sigma_1}^{\sigma_2} x_{ji}(t) = 0 \qquad \forall i \epsilon V\backslash\{s,d\}, \forall \sigma_1, \sigma_2\, \epsilon N \qquad (1)$$

$$0 \leq x_{ij}(t) \leq u_{ij}(t) \qquad \forall (i,j)\epsilon E\,,\ \ t\epsilon N \qquad (2)$$

$$\textstyle\sum_0^T x_{ji}(t) \leq U_{ij} \qquad \forall (i,j)\epsilon E \qquad (3)$$

$$x_{ij}(t) = 0 \qquad \forall (i,j)\epsilon E\ ,\forall\, t > T \qquad (4)$$

The value x_{ij}(t) is the amount of flow passing arc (i, j) at time step t . Conditions (1) are the flow conservation constraints for all intermediate nodes and time points t ϵ [0,T]. condition (2) represents the minimum and maximum possible amount of flow on arc (i, j) at any moment t. Condition (3) denotes the upper bound on total flow that we can send from node i along arc (i, j) during the time horizon T, not on the flow that becomes available on the arc (i,j) during the horizon T, in which we consider $\sum_0^T x_{ji}(t) \geq \mathbf{u}_{ij}$. Conditions (4) guarantee that the flow can just travel in the network until the end of a pre-specified time interval. We have assumed that no arcs enter the source node and no arcs leave the sink node. Such a flow is called a feasible discrete dynamic flow. Where x_{ij}(t) is the amount of flow passing through arc (i, j) at time step t. It is easy to observe that the flow does not enter arc

(i, j) at time step t if t ≥T.the value of the discrete dynamic s-d flow, x(t), is given by

$f = \sum_{t\epsilon N} \sum_{(s,j)\epsilon E} x_{sj}(t)$ (5)

Because of f is the total amount of flow leaving the source node s until time T, and that, because of flow conservation in intermediate nodes, this value is equal to the total amount of flow arriving into the sink node d until time T, i.e.

$\sum_j \sum_0^T x_{sj}(t) = \sum_j \sum_0^T x_{jd}(t)$

(6)

We can formulate the maximum dynamic flow problem on a discrete-time with time horizon T

$Max\, f$

$\sum_j \sum_0^T x_{sj}(t) = f$ (7)

$\sum_j \sum_0^T x_{jd}(t) = -f$ (8)

$\sum_j \sum_{\sigma_1}^{\sigma_2} x_{ij}(t) - \sum_j \sum_{\sigma_1}^{\sigma_2} x_{ji}(t) = 0 \qquad \forall i\epsilon V\backslash\{s,d\}, \forall \sigma_1, \sigma_2\, \epsilon[0,T],$ (9)

$\sum_0^T x_{ji}(t) \leq U_{ij} \qquad \forall (i,j)\epsilon E$ (10)

$0 \leq x_{ij}(t) \leq u_{ij}(t) \qquad \forall (i,j)\epsilon E, \; t\epsilon[0, \mathrm{T}]$ (11)

$x_{ij}(t) = 0 \qquad \forall (i,j)\epsilon E \;, \forall\, t > T$ (12)

Lemma 3.1. The value of any discrete dynamic s-d flow can be computed as $f = \sum_{t\epsilon N} f(t)$, where f(t) is the amount of flow leaving the source s at time step t ϵ T. Moreover,

$f(t) \leq \sum_{(s,j)\epsilon E} u_{sj}(t), \forall t\epsilon N \text{ and } f \leq \sum_{(s,j)\epsilon E} \mathbf{u}_{sj}.$

Proof- Since $\sum_j \sum_{\sigma_1}^{\sigma_2} x_{ij}(t) - \sum_j \sum_{\sigma_1}^{\sigma_2} x_{ji}(t) = \sum_{\sigma_1}^{\sigma_2}(\sum_j x_{ij}(t) - \sum_j x_{ji}(t)), \forall\, \sigma_1, \sigma_2 \epsilon N,$ we have $(\sum_j x_{ij}(t) - \sum_j x_{ji}(t)) = 0, \forall i\epsilon V\backslash\{s,d\}, \forall t\epsilon N.$ ■

3.2. The continuous time model

the maximum dynamic flow problem on a dynamic network with source s and sink d is the problem of finding a feasible flow with maximum value from source s to sink d within a pre-specified time horizon T such that flow going through arcs does not exceed their bounds. The feasibility constraints for a flow in this case are the same as discrete case. Hence, we can formulate the problem as follows:

Max f

$$\sum_j \int_0^T x_{sj}(t)d(t) = f \qquad (13)$$

$$\sum_j \int_0^T x_{jd}(t)d(t) = -f \qquad (14)$$

$$\sum_j \int_{\sigma_1}^{\sigma_2} x_{ij}(t)d(t) - \sum_j \int_{\sigma_1}^{\sigma_2} x_{ji}(t)d(t) = 0 \quad , \quad \forall i \epsilon V \backslash \{s,d\}, \forall \sigma_1, \sigma_2 \, \epsilon [0,T], \qquad (15)$$

$$\int_0^T x_{ji}(t)d(t) \leq \boldsymbol{U}_{ij} \quad , \quad \forall (i,j) \epsilon E \qquad (16)$$

$$0 \leq x_{ij}(t) \leq u_{ij}(t) \quad , \quad \forall (i,j) \epsilon E \, , \; t \epsilon [0, \mathrm{T}] \qquad (17)$$

$$x_{ij}(t) = 0 \quad , \quad \forall (i,j) \epsilon E \; , \forall \, t > T \qquad (18)$$

Lemma 3.2. The value of any dynamic s-d flow can be stated as $f = \int_0^T f(t)\, d(t)$ where f(t) is the amount of flow leaving the source s at time moment t ϵT. Moreover,

$f(t) \leq \sum_{(s,j)\epsilon E} u_{sj}\,(t), \forall t \epsilon [0,T]$ and $f \leq \sum_{(s,j)\epsilon E} \mathbf{u}_{sj}$.

Proof- certainly $f \leq \sum_{(s,j)\epsilon E} \int_0^T x_{sj}\,(t)dt = \int_0^T \sum_{(s,j)\epsilon E} x_{sj}\,(t)dt$. Let $(t) = \sum_{(s,j)\epsilon E} x_{sj}(t)$, then $f = \int_0^T f(t)dt$. Also we know that$x_{ij}(t) \leq u_{ij}(t), \forall (i,j) \epsilon E$, then

$f(t) \leq \sum_{(s,j)\epsilon E} u_{sj}\,(t)$. $f \leq \sum_{(s,j)\epsilon E} \boldsymbol{u}_{sj}$ is found by considering the horizon capacity.■

4. Generic augmenting path algorithm

To solve the formulated maximum dynamic flow problem we propose an approach based on transforming the problem to a minimum cost static flow problem. We consider the discrete-time models, in which all times are integral and bounded by T. We show that the dynamic problem on such network flow G= (V, E, T) can be reduced to a static flow problem on an auxiliary network $G^T = (V^T, E^T)$, called time expanded network. The advantage of this method is that it transforms the problem of determining a maximum flow on a dynamic network flow into a static network flow problem. We first give some definitions and notation that we will use in rest of the paper.

Definition 1. *Dynamic cut:* If $S \subset V$, $S \neq \emptyset$ and $\hat{S}=V \setminus S$ then a dynamic cut $(S, \hat{S})$, is defined as

$(S, \hat{S})= \{(i,j) \epsilon E : i \epsilon S , j \epsilon \hat{S} \}$

Definition 2. *s-d dynamic cut*: If the set S contains the source ***s*** and the set $\hat{S}$ contains the consumer sink ***d*** then a dynamic cut $(S, \hat{S})$ is a **s-d** dynamic cut.

Definition 3. The capacity of a dynamic cut $(S, \hat{S})$ is defined as: $u(S,\hat{S}) = \sum_{(i,j)\epsilon(S,\hat{S})} \mathbf{u}_{ij}$

Clearly, the capacity of a cut is an upper bound on the maximum amount of flow we can send from the nodes in S to the nodes in $\hat{S}$ while honoring arc flow bounds.

Definition 4. *Residual Dynamic Network*. We define the residual dynamic network with respect to a given feasible dynamic flow $x(t)$ as $G(x(t))=(V,E_1^*,E_2^*,T)$ where

$$E_1^* = \{(i,j)\epsilon E: u_{ij} \geq \textstyle\sum_{t\epsilon N} x_{ij}\,(t) \tag{19}$$

$$E_2^* = \{(i,j)\epsilon E: 0 < \textstyle\sum_{t\epsilon N} x_{ij}\,(t) \tag{20}$$

The residual dynamic network $G(x(t))$ is provided with residual arc capacities as:

$$u_{ij}^* = \mathbf{u}_{ij} - \textstyle\sum_{t\epsilon N} x_{ij}\,(t) \quad ,(i,j)\epsilon E_1^* \tag{21}$$

$$u_{ji}^* = \textstyle\sum_{t\epsilon N} x_{ij}\,(t) \qquad ,(i,j)\epsilon E_2^* \tag{22}$$

Theorem 1. (*Maximum Dynamic Flow-Minimum Dynamic Cut*). The value of a maximum dynamic flow from the source node ***s*** to the sink node ***d*** in a dynamic network equals the value of a s-d dynamic cut with minimum capacity. [2]

Definition 5. *Time-expanded Network*: Ford and Fulkerson introduce the notion of time-expanded networks. A time-expanded network contains one copy of the node set of the underlying 'static' network for each discrete time step (building a time layer). For a dynamic network $G= (V,E,T)$ the time expanded network $G^T= (V^T,E^T)$ is defined as follows: A *time-expanded network* of G, denoted by $G(\varphi)$,where $\varphi = \{t_0, t_1, \dots, t_p\}$ contains p+1 copies of V, denoted by $V_0, V_1, \dots, V_p$,in which V_{q-1} corresponds to the time step t_{q-1} for $q = 1, \dots, p-1$, and V_p to the time horizon T. Subsequently, index q varies from 1 to p. The copy of node $i \epsilon V$ in V_{q-1} is denoted by i_{q-1}. For each arc $(i,j) \epsilon E$ and each time $t_{q-1} \epsilon \varphi$ with $0 \le t_{q-1} + \lambda_{i,j} \le T$, Traversing through arc$(i_{q-1}, j_{q'})$ where, $t_{q'} = t_{q-1}+\lambda_{i,j}$ corresponds to leaving node i at time t_{q-1} and arriving at node j at time $t_{q'}$. Hence, arc (i_{q-1}, j_q) has an associated cost $c_{i,j}(t_{q-1})$. For each node i, there is a holdover arc from i_{q-1} to i_q . Traveling through arc (i_{q-1}, i_q) corresponds to the parking at node i from time t_{q-1} to t_q. So holdover arc (i_{q-1}, i_q) has an associated cost $f_i(t_{q-1})$. [14]

Now, with regard to the definition of time expanded network we get the following equivalent formulation of the maximum dynamic flow problem on a generative network G.

$$\max f^T = \sum_{t \epsilon N} f^t$$

$$\sum_j x_{ij}^t - x_{ji}^t = f^t \ , i = s^t \quad , \quad t\epsilon\varphi \qquad (23)$$

$$\sum_j x_{ij}^t - x_{ji}^t = 0 \ , i \neq s^t, d^t \quad , \quad t\epsilon\varphi \qquad (24)$$

$$\sum_j x_{ij}^t - x_{ji}^t = -f^t \ , i = d^t \quad , \quad t\epsilon\varphi \qquad (25)$$

$$0 \le \sum_{t\epsilon\varphi} x_{ij}^t \le \mathbf{u}_{ij}^T \ , \ \{(i,j)^t | t\epsilon\varphi\} \ , \ (i,j)\epsilon E \qquad (26)$$

$$0 \le x_{ij}^t \le u_{ij}^t \ \forall (i,j)^t \epsilon E^T \quad , \quad t\epsilon\varphi \qquad (27)$$

After transforming dynamic network to static network we use the most intuitive algorithms for solving the maximum flow problem. This algorithm is known as the *augmenting path algorithm.* The generic augmenting path algorithm is essentially based on this simple observation. The algorithm proceeds by identifying augmenting paths and augmenting flows on these paths until the network contains no such path. Following algorithm describes the generic augmenting path algorithm:[2]

algorithm *augmenting path;*

begin

x: = 0;

while *G(x)* contains a directed path from node ***s*** to node ***d*** **do**

begin

identify an augmenting path *P* from node ***s*** to node ***d****;*

β := min{r_{ij} : (i, j) ϵ P};

augment β units of flow along *P* and update *G(x);*

end;

end;

We use the maximum flow problem given in Figure1 (a) to illustrate the algorithm. Suppose that the algorithm selects the path 1-3-4 for augmentation. The residual capacity of this path is β = $min\ \{r_{13}, r_{34}\}$ = min {4, 5} = 4. This augmentation reduces the residual capacity of arc (1, 3) to zero (thus we delete it from the residual network) and increases the residual capacity of arc (3, 1) to 4 (so we add this arc to the residual network). The augmentation also decreases the residual capacity of arc (3, 4) from 5 to 1 and increases the residual capacity of arc (4, 3) from 0 to 4. Figure1 (b) shows the residual network at this stage. In the second iteration, suppose that the algorithm selects the path 1-2-3-4. The residual capacity of this path is 8 = min {2, 3, I} = 1. Augmenting 1 unit of flow along this path yields the residual network shown in Figur1 (c). In the third iteration, the algorithm augments 1 unit of flow along the path 1-2-4. Figure1 (d) shows the corresponding residual network. Now the residual network contains no augmenting path, so the algorithm terminates.

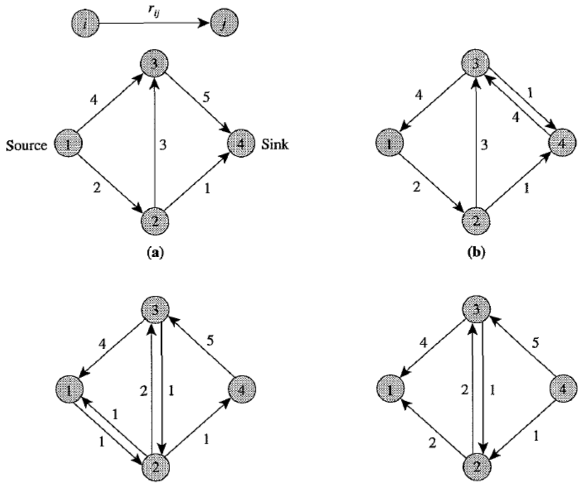

Figure1.Illustrating the generic augmenting path algorithm: (a) residual network for the zero flow; (b) network after augmenting four units along the path 1-3-4; (c) network after augmenting one unit along the path 1-2-3-4; (d) network after augmenting one unit along the path 1-2-4.[2]

5. Conclusion

In this literature, we introduced a type of network flows, called dynamic network flows which the flow commodity is dynamically generated in a source node **s** and consumed in a sink node **d**. The mathematical models of the maximum dynamic flow problem in for discrete and continuous time were built and analyzed. The dynamic cuts flows were defined. Moreover, by transforming the dynamic network flow to a static network, called time expanded network, we showed that the problem can be solved with the generic augmenting path algorithm.

References

[1]. Aronson, J. E.: A survey of dynamic network flows, *Ann. Oper. Res*. **20** (1989), 1–66.

[2]. Ahuja, R.K., Orlin, J.B., Pallottino, S., Scutella, M.G.: Dynamic Shortest Paths Minimizing Travel Times and Costs. Networks **41**, 197–205 (2003)

[3]. Cai, X., Kloks, T., Wong, C.K.: Time-varying shortest path problems with constraints. Networks **29**, 141–149 (1997)

[4]. Chabini, L.: Discrete dynamic shortest path problems in transportation applications: Complexity and algorithms with optimal run time. Transp. Res. Rec. **1645**, 170–175 (1998)

[5]. Orda, A., Rom, R.: Shortest-path and minimum-delay algorithms in networks with time-dependent edge length. J. Assoc. Comput. Mach. **37**, 607–625 (1990)

[6]. Orda, A., Rom, R.:Minimum weight paths in time-dependent networks. Networks **21**, 295–320 (1991)

[7]. Pallottino, S., Scutella, M.G.: Shortest path algorithms in transportation models: Classical and innovative aspects. In: Marcotte, P., Nguyen, S. (eds.) Equilibrium and advanced transportation modelling, pp. 245–281. Kluwer, Norwell (1998)

[8]. Philpott, A.B., Mees, A.I.: Continuous-time shortest path problems with stopping and starting costs. Appl. Math. Lett. **5**, 63–66 (1992)

[9]. Philpott, A.B., Mees, A.I.: A finite-time algorithm for shortest path problems with time-varying costs. Appl. Math. Lett. **6**, 91–94 (1993)

[10]. Pilpoitt, A.B.: Continuous-time shortest path problems and linear programming. SIAM J. Control Optim. **32**, 538–552 (1994)

[11] W. Powell, P. Jaillet, A. Odoni, Stochastic and dynamic networks and routing, in: M.O. Ball, T.L. Magnanti, C.L. Monma, G.L. Nemhauser (Eds.), Network Routing, Handbooks in Operations Research and Management Science, vol. 8, North–Holland, Amsterdam, The Netherlands, 1995, pp. 141–295. Chapter 3.

[12]. B. Hoppe, Efficient Dynamic Network Flow Algorithms, Ph.D. Thesis, Cornell University, 1995.

[13]. M. Fonoberova, D. Lozovanu, Optimal dynamic flows in networks and applications, in: The International Symposium on the Issues of Calculation Optimization, Communications, Crimea, Ukraine, 2007, pp. 292–293.

[14]. L. Ford, D. Fulkerson, Flows in Networks, Princeton University Press, Princeton, NJ, 1962.