A LOGISTIC BROWNIAN MOTION WITH A PRICE OF DIVIDEND YIELDING ASSET

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Abstract:

In this paper, we have used the idea of Onyango (2003) he used to develop a logistic equation used in natural science to develop a logistic geometric Brownian motion with a price of dividend yielding asset since in reality, assets do pay dividends to shareholders at regular intervals after a maturity period which normally after one year. We have first derived a deterministic Walrasian price-adjustment model with price of dividend yielding asset after which we have introduced the stochastic effect of noise to derive a logistic geometric Brownian motion with a unique volatility function $\sigma(S,t)$ such that the observed option price is consistent with the price of the dividend yielding asset model.

Keywords: Volatility, Modeling, Geometric Brownian motion, Supply and Demand functions, Equilibrium price, Walrasian excess demand function, deterministic Walrasian price-adjustment model, Logistic Brownian motion.

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1. Introduction:

The supply and demand curves determine the price and quantity at which assets are bought and sold. The demand curve shows the quantities buyers want to buy at various prices and the supply curve shows the quantities the sellers want to sell at various prices. A situation in which there is no tendency for change in security price and security quantity at a given point in time is known as market equilibrium. That is, there is no reason for the market price of products to rise or fall. In stock markets, the price of an asset is assumed to respond excess demand and is expressed as;

$$\text{ED}\, S(t) = Q_D S(t) - Q_S S(t) \qquad 1.1$$

Where ED S(t) is the excess demand, $Q_D S(t)$ and $Q_S S(t)$ are quantities demanded and supplied respectively at a given time , t, and price S(t). The logistic equation was first used by Verlhust (1838) and Reed (1920) in population growth. In Verlhust's model for studying dynamics of human population growth in United States, he took P*represent the environmental carrying capacity in which a population lives, which favorably compares to S* the Walrasian equilibrium marked price, a price at which the market supply and demand are equal. Onyango (2003) using Verhlusts logistic equation developed a logistic equation for asset security prices, considering the fact that naturally asset security prices would not usually shoot indefinitely (exponentially) due to a regulating factor that may limit the asset prices. He introduced the excess demand functions and applied them in the Walrasian (Walrasian Samuelson) price adjustment mechanism to obtain a non-linear Brownian motion of the form

$$dS(t) = \mu S(t)(S^* - S(t))dt + \sigma S(t)(S^* - S(t))dZ(t) \qquad 1.2$$

Where S* is the equilibrium marked price, S(t) is the market price of the asset or security, μ is the rate of increase of the asset and σ is the volatility while dZ(t) is the standard wiener process.

2. Preliminaries:

The law of demand

The law of demand gives the relationship between demand price of an asset and quantity demanded *(ceteris paribus*). The law states that as prices of product decreases, the quantity of the product that buyers are able and willing to purchase in a given period of time increases if other factors remain unchanged *(ceteris paribus.)*

The law of supply

The law of supply indicates the relationship between supply price and quantity supplied *(Ceteris paribus).* That is suppliers will supply less of goods or services at low price and as price rises the quantity supplied will increase if other factors remain unchanged *(ceteris paribus.)*

Equilibrium price (Walrasian Price Equilibrium)

This is a state of stability or balance where the quantity of a good or service supplied is equal to the quantity of the same good or service demanded. This state leads to equilibrium price S* and equilibrium quantity Q*. That is

$$Q_D(S^*) = Q_S(S^*) \quad \text{and} \quad Q_D(Q^*) = Q_S(Q^*) \qquad 2.1$$

where $Q_D(Q^*)$ the quantity is demand functions and $Q_S(Q^*)$ is quantity supply functions. Walrasian price equilibrium states that the total demand does not exceed total supply and vice versa.

3. Wiener Process (Brownian motion):

Wiener process is a particular type of Markov Stochastic process with mean change of zero and variance 1.0. If X (t) follows a stochastic process where μ the mean of the probability distribution is and σ is the standard deviation. That is, X (t) ~ N (μ, σ) then for Wiener process

X (t) ~ N (0, 1) which means X (t) is a normal distribution with μ=0 and σ=1. Expressed formally, a variable Z follows a Wiener process if it has the following properties:

PROPERTY 1: The change ΔZ during a small period of time i

$$\Delta Z = \varepsilon \sqrt{\Delta t} \tag{3.1}$$

where ε has a standardized normal distribution; $\vartheta(0,1)$

PROPERTY 2: The values of ΔZ for any two different short time intervals of time, Δt, are independent. That is, Var (ΔZ_i, Z_j)=0, i $\neq$ j. It follows from the first property that itself has a normal distribution with Mean of ΔZ=0, Standard deviation of ΔZ= $\sqrt{\Delta t}$ and Variance of ΔZ= Δt. The second property implies that Z follows a Markov process. Consider the change in the value of Z during a relatively long period of time T. This can be denoted by Z (T)-Z (0). It can be regarded as the sum of the changes in Z in N small time intervals of length Δ (t), where

$$\frac{T}{\Delta t}$$

Thus $$Z(T)-Z(0) = \sum_{i=1}^{N} \varepsilon_i \sqrt{\Delta t}, \tag{3.2}$$

where the ε_i (i=1,2,3.....N) are distributed $\vartheta(0,1)$. From the second property of Weiner process, ε_i are independent of each other. It follows that Z (T)-Z (0) is normally distributed with Mean =E (Z (T)-Z (0)) = 0, Variance of (Z (T)-Z (0)) = nΔ= T thus, Standard deviation of (Z (T)-Z (0)) is $\sqrt{T}$ Hence Z (T)-Z (0) ~ N (0, $\sqrt{T}$).

It$^\wedge$o process is a generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable X and time t. An It$^\wedge$o process can be written algebraically as,

$$dX = a(X, t)\, dt + b(X, t)\, dZ. \tag{3.3}$$

It$^\wedge$o lemma is the formula used for solving stochastic differential equations. It is a treatment of wide range of Wiener-like differential process into a strict mathematical framework.

4. Geometric Brownian motion:

A specific type of Ito's process is the geometric Brownian motion of the form

$$dX=aXdt+bXdZ \qquad 4.1$$

where a(X,t)=aX and b(X,t) =bX (samuelson, 1965 Black and Scholes,1973) The geometric Brownian motion has been applied in stock pricing and is given as

$$dS = \mu S dt + \sigma S dZ \qquad 4.2$$

Where S is the stock price, μ is the expected rate of return per unit time and σ is the volatility of stock price.

5. Deterministic Walrasian price-adjustment model:

Onyango (2003) borrowed from the Walrasian –Samuelson model core principle that sellers call out prices of commodities and there is no immediate trade until excess demand is zero and brought in the idea that security prices are related to excess demand

If we consider fractional rate of increase of asset price $\frac{1}{S(t)} \frac{dS(t)}{dt}$, it can be shown to be proportional to excess demand function, hence we have

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = Q_D S(t) - Q_s S(t) \qquad 5.1$$

The general expression of a linear demand function can be given as

$$Q_D S(t) = -aS(t) + b \qquad 5.2$$

Where S(t) is the price of asset at time t and S* is the equilibrium price for some appropriate parameters a and b.

Similarly a linear supply function is expressed as

$$Q_S S(t) = cS(t) - d \qquad 5.3$$

For some appropriate parameters c and d (Jacques 1992)

A linearise demand and supply curves about equilibrium, gives equations (5.2) and (5.3) as

$$Q_D(S(t) = \alpha(S^* - S(t)) \text{ and } Q_S(S(t) = -\beta(S^* - S(t)) \qquad 5.4$$

Where α is the demand elasticity and β is the supply elasticity. Equation (5.4) requires that $Q_D S(t)$ is a decreasing linear function of $S(t), \alpha > 0$ and $Q_S S(t)$ is an increasing linear function of $S(t), \beta < 0,$ The excess demand function is given as

$$EDS(t) = Q_D S(t) - Q_S(t) = (\alpha + \beta)(S^* - S(t)) \qquad 5.5$$

From proportional rate of increase in equation (5.1) and excess demand function in equation (5.5) we have

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = h(\alpha + \beta)(S^* - S(t)) \qquad 5.6$$

Or

$$\frac{dS(t)}{dt} = \mu S(t)(S^* - S(t)) \qquad 5.7$$

Or

$$dS(t) = \mu S(t)(S^* - S(t)) \qquad 5.8$$

Where $\mu = h(\alpha + \beta)$ and is a growth constant. The asset price growth rate is also equal to zero when S(t)=o. This is a deterministic logistic (first order) ordinary differential equation in stock price S(t) and the limiting constant S(t) is also known as Verhlust logistic equation.

In reality, assets do pay dividends to Shareholders. In this case we consider payments of dividends to be continuous. The Verhlust logistic equation in (5.7) becomes

$$\frac{dS(t)}{dt} = (\mu - q)S(t)(S^* - S(t)) : \mu - q = h(\alpha + \beta) \qquad 5.9$$

Where q is the dividend paying rate.

Or $$\frac{dS(t)}{S(t)(S^{*}-S(t)}=(\mu-q)d(t) \quad 5.10$$

The L.H.S of equation (5.10) can be done using Heaveside cover up method, letting $S(t)=0$ and $S^{*}=S(t)$ after integrating one has

$$\frac{1}{S^{*}}\ln\left|\frac{S(t)}{S^{*}-S(t)}\right|_{t_o}^{t}=(\mu-q)t\Big|_{t_0}^{t} \quad 5.11$$

So $$In\left|\frac{S(t)}{S^{*}-S(t)}\right|=In\left|\frac{S(t_0)}{S^{*}-S(t_0)}\right|+(\mu-q)S^{*}(t-t_0) \quad 5.12$$

Rearranging and simplifying equation (5.12) and solving for S(t) we get

$$S(t)=\frac{S^{*}(S(0))}{S(0)+(S^{*}-S(0)e^{-(\mu-q)S^{*}(t-t_0))}}, \qquad \text{if } P(t_0)=P(0) \quad 5.13$$

This is a deterministic logistic equation in asset price S(t), with initial price S(0), equilibrium price S*, μ asset price growth rate and q dividend yield rate.

6. Logistic geometric Brownian motion model with dividend yielding asset:

From equation (5.6) we extend Walrasian Price adjustment model by introducing actual asset prices with noise effect which is a Weiner process. We consider trading ground where bargains gives small random effect in sensitivities of α and β in supply and demand functions. We let $\delta\alpha$ and $\delta\beta$ be accumulative changes in price sensitivities of α and β respectively during trading period δt. Then equation (5.6) becomes.

$$\frac{1}{S(t)}\frac{dS(t)}{dt}=h(\alpha+\delta\alpha+\beta+\delta\beta)(S^{*}-S(t))$$

$$=h(\alpha+\beta)(S^{*}-S(t))+h(\delta\alpha+\delta\beta)(S^{*}-S(t)) \quad 6.1$$

Wiener process on supply and demand from cumulative effects on random shock on $\delta\alpha$ and $\delta\beta$ is put as

$$h(\delta\alpha+\delta\beta)=\sigma dZ \qquad 6.2$$

Where σ the standard deviation (volatility) of underlying asset, Z is is the standard Wiener process ($Z=\varepsilon\sqrt{dt}$, $\varepsilon \rightarrow N(0,1)$

Substituting (6.2) in (6.1) we have

$$\frac{1}{S(t)}\frac{dS(t)}{(S^{*}-S(t))}=(\mu-q)dt+\sigma dZ \qquad P^{*}\neq P(t) \qquad 6.3$$

Using the same process for solving (5.13). Equation (6.3) becomes

$$In\left|\frac{S(t)}{S^{*}-S(t)}\right|=In\left|\frac{S(t_0)}{S^{*}-S(t_0)}\right|+(\mu-q)S^{*}(t-t_0)+\sigma S^{*}dZ(t) \qquad 6.4$$

Rearranging and simplifying equation (6.4) we have

$$S(t)=\frac{S^{*}(S(0))}{S(0)+(S^{*}-S(0)e^{-(\mu-q)S^{*}(t-t_0)+\sigma S^{*}dZ(t))}}, \qquad 6.5$$

This is logistic Brownian motion model in stock price with dividend yielding asset.

7. Conclusion:

In this paper we have developed logistic geometric Brownian motion with a price of dividend yielding asst since in reality assets do pay dividend to shareholders. This can be used to show a unique volatility function $\sigma(S,t)$ from observed option price which is consistent with dividend yielding asset model in (6.5)

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